LIE GROUPS: HISTORY, FRONTIERS AND APPLICATIONS

VOLUME XV

The Classical Differential Geometry of Curves and Surfaces

GEORGES VALIRON

Translated by James Glazebrook

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by Robert Hermann

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PREFACE

by Robert Hermann

This translation by Jim Glazebrook completes Volume 2 of Valiron's *Cours d'Analyse*. Refer to the Preface of *Lie Groups*, Volume XIV, for my reasons for undertaking this project.

As pure and applied differential geometry gets ever fancier and more complicated, it is well to keep in mind where it all came from: Darboux's *Theory des Surfaces*. However, there is nowhere in the English-language literature where a student can find an adequate explanation of this "classical" point of view. (The only counterexamples to this statement are Eisenhart's treatises. But they are a pale rendering of the original.) When I was a student in the 1950s, I found Valiron's distillation of this material to be most helpful. I hope Glazebrook's translations will help the present generation to rediscover this Mother Lode of geometric wisdom.

This is probably the last of the series of Translations. I started over ten years ago with the seminal papers of Lie, Ricci, and Levi-Civita, and have fulfilled the goal that I set myself: To make available in English, with some commentaries, the classics that I see as the underpinnings of today's pure and applied geometric world.

I would like to thank Jim Glazebrook again for his work in these translations of Cartan and Valiron. Renate D'Arcangelo and Karin Young have typed this manuscript; I thank them also.

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Chapter XII

THE THEORY OF SPACE CURVES

Differential geometry developed concurrently with analysis. The problems relating to the curvature of plane curves were approached by Newton, Leibniz, and Huygens; the curvature of surfaces was studied by Euler. The theory of envelopes of planes or of surfaces, commenced toward the end of the 18-th century. In 1785, Monge introduced the notion of the polar surface and the envelope of the normal planes to a skew curve. In 1805, Lancret established the properties of the rectifying line, the line of contact of the rectifying plane with its envelope; this is his own terminology. The theory of skew curves as developed in this chapter, dates from around 1850; the mode of exposition in terms of vectorial derivatives is more recent and makes no essential difference.

The formulae of Frenet and Serret were established between 1847 and 1851; the binormal denomination was due to Barré de Saint-Venant; the theorems of Puiseaux and Bertrand are those established between 1851 and 1858.

More recently, the development of topology brought about a different and more general perspective on problems in algebraic geometry, relating to intersections and tangents. Having established Cayley's formulae in the most general case, we will remark on "the finite geometry" of Darboux. Likewise, we shall say something about direct differential geometry, which introduces the methods of the theory of functions of real variables into the theory of curves and surfaces.

In everything relating to differential geometry, we shall restrict our attention to three dimensional space, and unless otherwise stated, we shall take the data to be real.

I. THE FORMULAE OF FRENET-SERRET -- APPLICATIONS

204. Recalling the known results

We have already encountered space curves in several situations, and have discussed some of their properties. One such curve C is defined either by the coordinates of its varying point in terms of functions of a parameter

$$x = f(u), \qquad y = g(u), \qquad z = h(u) \tag{1}$$

or, by the extremity M of a vector tied to the origin and a function of u,

$$\vec{M} = \overrightarrow{M(u)}$$

The tangent at the point M is defined by the derivative vector from M when this vector exists and is nonzero; hence by

$$\frac{d\vec{M}}{du} \quad \text{or} \quad x' = f'(u), \quad y' = g'(u), \quad z' = h'(u)$$

Unless specified to the contrary, we shall assume that the axes are rectangular. If we assume that the arc passing through M when u varies from u_0 to u corresponds bijectively to the segment (u_0,u) and if the tangent vector is continuous, then this will have a length (I,51). If we measure the arc positively in the sense of u increasing, then its algebraic value s is given by

$$s = \int_{u_0}^{u} \left|\frac{d\vec{M}}{du}\right| du = \int_{u_0}^{u} \sqrt{x'^2 + y'^2 + z'^2}\, du$$

The unit vector of the tangent in the sense of u increasing is

$$\vec{t} = \frac{d\vec{M}}{ds} = \frac{d\vec{M}}{du} \cdot \frac{du}{ds}$$

$$\frac{ds}{du} = \sqrt{x'^2 + y'^2 + z'^2} = \delta$$

its components are

$$\alpha = \frac{dx}{ds} = \frac{x'}{\delta}, \quad \beta = \frac{dy}{ds} = \frac{y'}{\delta}, \quad \gamma = \frac{dz}{ds} = \frac{z'}{\delta}$$

In all that follows, we shall consider an arc *without turning points*. When the curve is given by the intersection of two surfaces, we know how to bring it to the form in (1) and to determine its tangent (I,124).

We have explained (no. 41) what is meant by an analytic curve and by an ordinary point of such a curve. In all cases, we have seen from no. 56 how to apply the Taylor formula to a vector. In no. 57 we defined the osculating plane. In the general case, it is the plane defined by the vectors

$$\frac{d\vec{M}}{du}, \quad \frac{d^2\vec{M}}{du}$$

THE INDICATRIX OF THE TANGENTS

This is the locus Γ of the extremity μ of the vector equipollent to $\vec{t}$ and having as its origin, a fixed point that can be placed at 0. It is therefore a spherical curve described on a sphere of radius 1. When we change

the sense of a positive direction on the curve C, then the indicatrix Γ is replaced by curve symmetric with respect to O. When the second derivative vector of $\vec{M}$ exists and is continuous, then Γ is an arc. If we take as the positive sense on Γ, the sense of the displacement of μ when u increases, then the unit vector corresponding to the tangent to the indicatrix is

$$\vec{n} = \frac{d\vec{t}}{d\sigma} = \frac{d\vec{t}}{ds}\frac{ds}{d\sigma} = \frac{d^2\overrightarrow{M}}{ds^2}\frac{ds}{d\sigma} \tag{2}$$

if σ is taken to be the arc of the indicatrix. The *radius of curvature* is the positive number

$$R = \frac{ds}{d\sigma}$$

The relationship (2) is then written as

$$\frac{d^2\overrightarrow{M}}{ds^2} = \frac{d\vec{t}}{ds} = \frac{\vec{n}}{R}, \qquad R = \frac{ds}{d\sigma} \tag{3}$$

The vector $\vec{n}$ is clearly normal to the curve C at the point M. It defines *the principal normal* (it is, in fact, a semi-normal). We can easily verify that the sense of this principal normal does not change when the sense of the positive direction is changed on C. But this property will also result in the following result:

The principal normal is contained in the osculating plane.

205. The trihedron of Frenet-Serret. Torsion. Formulae of Frenet-Serret.

Let us consider the trihedron defined by the positive tangent at M, the principal normal (semi-normal) and the semi-normal that forms with these first two half-lines, a trirectangular trihedron. The unit vector of this last direction, known as a *binormal*, will be denoted by b. We then have

$$\vec{b} = \vec{t} \wedge \vec{n}, \qquad \vec{n} = \vec{b} \wedge \vec{t}, \qquad \vec{t} = \vec{n} \wedge \vec{b}$$

The trihedron thus defined is the trihedron of Frenet-Serret. If we change the sense of the positive direction on C, then the trihedron is replaced by one symmetric with respect to the principal normal. The sense of the binormal is changed in the manner of that of the positive tangent.

TORSION

When we differentiate the relationship $b = t \wedge n$ with respect to s, we obtain:

$$\frac{d\vec{b}}{ds} = \vec{t} \wedge \frac{d\vec{n}}{ds} + \frac{d\vec{t}}{ds} \wedge n$$

$$= \vec{t} \wedge \frac{d\vec{n}}{ds}$$

since the second term in the second member is zero on account of (3). Now,

$$\vec{n}^2 = 1 , \qquad \vec{n} \cdot \frac{dn}{ds} = 0$$

shows that

$$\frac{d\vec{n}}{ds} = k\vec{t} + k'\vec{b} \quad ,$$

hence

$$\frac{d\vec{b}}{ds} = -\frac{\vec{n}}{t} \tag{4}$$

The number $1/T$, which is defined by this relationship, is known as the *torsion*. It has a sign. Its definition implies the existence of the third derivative of $\vec{M}$. We also say that T is the radius of torsion.[1]

<u>Formulae of Frenet-Serret</u>. Differentiating the relationship

$$\vec{n} = \vec{b} \wedge \vec{t}$$

yields, on account of the relationships (3) and (4) already obtained,

$$\begin{aligned} \frac{d\vec{n}}{ds} &= \frac{d\vec{b}}{ds} \wedge \vec{t} + \vec{b} \wedge \frac{d\vec{t}}{ds} \\ &= -\frac{1}{T}\vec{n} \wedge \vec{t} + \frac{1}{R}\vec{b} \wedge \vec{n} \\ &= -\frac{\vec{t}}{R} + \frac{\vec{b}}{T} \end{aligned}$$

The set of formulae (3) and (4) together with the one just obtained, constitutes the formulae of Frenet-Serret:

$$\begin{aligned} &\frac{d\vec{t}}{ds} = \frac{\vec{n}}{R} , \qquad \frac{d\vec{n}}{ds} = -\frac{\vec{t}}{R} + \frac{\vec{b}}{T} \\ &\frac{d\vec{b}}{ds} = -\frac{\vec{n}}{T} \end{aligned} \tag{5}$$

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[1]The sign taken for T is in acccordance with Darboux's convention.

The Taylor expansion of the vector M. When we give s an increment s, we obtain

$$\vec{M}(s+\Delta s) = \vec{M} + \Delta s \frac{d\vec{M}}{ds} + \frac{\Delta s^2}{2} \cdot \frac{d^2\vec{M}}{ds^2} + \frac{\Delta s^3}{6} \frac{d^3\vec{M}}{ds^3} + \vec{0}(\Delta s^2)$$

which on taking into account the Frenet formulae yields

$$\vec{M}(s+\Delta s) = \vec{M} + \Delta s\vec{t} + \frac{\Delta s^2}{2} \frac{\vec{n}}{R} + \frac{\Delta s^3}{6} \left[-\frac{\vec{t}}{R^2} + \vec{n} \frac{d}{ds} \cdot \frac{1}{R} + \frac{\vec{b}}{RT} \right] + \vec{0}(\Delta s^3)$$

If we denote the components on the axes Mt, Mn, and Mb of the trihedron of Frenet-Serret of the vector joining M to the neighboring point $M(s+\Delta s)$, by X,Y and Z, respectively, then we have

$$X = \Delta s - \frac{\Delta s^3}{6} \frac{1}{R^2} + 0(\Delta s^3)$$

$$Y = \frac{\Delta s^2}{2} \frac{1}{R} - \frac{\Delta s^3}{6} \frac{R'}{R^2} + 0(\Delta s^3)$$

$$Z = \frac{\Delta s^3}{6} \frac{1}{RT} + 0(\Delta s^3)$$

It follows that, for $|\Delta s|$ small, the curve is on the same side of the plane $y = 0$ as the principal normal $(Y > 0)$, as it crosses the osculating plane (Z changes sign when s is zero). The first property does in fact show that the sense of the principal normal is independent of the parametric representation. The projections of the curve onto the planes Mtn (osculating plane), Mnb (normal plane), and Mbt (*rectifying* plane) introduce at M, respectively: an ordinary point, a turning point, and a point of inflexion. Clearly, R and T are assumed to be finite.

From these formulae we can immediately deduce that, for M' the neighboring point of M corresponding to the value $s+\Delta s$ of the arc and P its projection onto the tangent, we have

$$|\vec{MP}| = |\Delta s| + 0(\Delta s),$$

$$|\vec{M'P}| = \frac{\Delta s^2}{2R} + 0(\Delta s^2) \quad ,$$

hence

$$R = \lim_{\Delta s = 0} \frac{\overrightarrow{MP}^2}{2|\overrightarrow{M'P}|}$$

It follows that the circle tangent to the curve C at M and passing through tne neighboring point M' has as its limit the osculating circle at M when M' tends toward M. This osculating circle was introduced in no. 57 along with the osculating sphere. The osculating circle is the section of the osculating sphere by the osculating plane (no. 57); the center of the osculating sphere has coordinates $0, R, TR', R'$, with respect to the Frenet trihedron, where R' is the derivative of R with respect to the arc s.

INDICATRIX OF THE BINORMALS

This is the locus of the extremity ν of the vector equipollent to $\vec{b}$, whose origin is a fixed point. We shall take this fixed point to be identified with the point 0, which will serve to define the indicatrix of the tangents. Let Γ' be this indicatrix. At the points μ and ν of the two indicatrices corresponding to a point M of Γ, the tangents are parallel (on account of the first and the last formula of Frenet-Serret). The cones with vertices 0 and bases Γ' and Γ are supplementary. If ε is the angle of the tangents to C at the points $M(s)$ and M' corresponding to $s + \Delta s$, and ε' the angle of the binormals, then following the same Frenet formulae, we have

$$R = \lim_{\Delta s = 0} \frac{|\Delta s|}{\varepsilon} , \qquad |T| = \lim_{\Delta s = 0} \frac{|\Delta s|}{\varepsilon}$$

Remark. The binormal is normal to C at the point M and at the infinitely near point $M'(s + ds)$, since, for this point we have

$$\vec{t} + d\vec{t} = \vec{t} + \frac{\vec{n}}{R} ds$$

and $\vec{b}$ is orthogonal to $\vec{t}$ and $\vec{n}$, hence to $\vec{t} + d\vec{t}$.

206. Calculating the radii of curvature and torsion.

If u is regarded as representing time, then the vectors

$$\vec{V} = \nu\vec{t} = \frac{d\vec{M}}{du} , \qquad \vec{J} = \frac{d^2M}{du^2}$$

are the speed and the acceleration respectively, where ν is the algebraic measure of the speed

$$\nu = \frac{ds}{du} .$$

We have

$$\vec{J} = \frac{dv\vec{t}}{du} = \vec{t}\,\frac{dv}{du} + v\,\frac{d\vec{t}}{du}$$

$$= \vec{t}\,\frac{dv}{du} + v\,\frac{d\vec{t}}{ds}$$

$$= \vec{t}\,\frac{dv}{du} + \frac{v^2}{R}\,\vec{n} \tag{6}$$

which states that the acceleration is the sum of the normal acceleration and of the tangential acceleration. It follows that

$$\vec{V} \wedge \vec{J} = \frac{d\vec{M}}{du} \wedge \left(\vec{t}\,\frac{dv}{du} + \frac{\nu^2}{R}\,\vec{n}\right)$$

$$= \frac{\nu^3}{R}\,\vec{t} \wedge \vec{n}$$

$$= \frac{\nu^3}{R}\,\vec{b} \tag{7}$$

Hence

$$R = \frac{|\nu^3|}{|\vec{V} \wedge \vec{J}|}$$

Consequently, if the curve C is given by the formulae in (1), then in rectangular axes, we have

$$R = \frac{(x'^2 + y'^2 + z'^2)^{3/2}}{[(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2]^{\frac{1}{2}}}$$

On differentiating $\vec{J}$, we have on account of (6),

$$\frac{d\vec{J}}{du} = \frac{\vec{b}\nu^3}{RT} + k\vec{n} + k''\vec{t}$$

where k and k' are functions of s that need not be calculated. Hence, by scalar multiplying this relationship by the extreme members of (7), we will obtain the following relationship where the first member is a mixed product

$$\left(\frac{d\vec{M}}{ds},\ \frac{d^2\vec{M}}{du^2},\ \frac{d^3\vec{M}}{du^3}\right) = \frac{\nu^6}{R^2T} = \frac{1}{T}\left(\frac{d\vec{M}}{du} \wedge \frac{d^2\vec{M}}{du^2}\right)$$

We then have

$$T = \frac{\left(\dfrac{d\vec{M}}{du} \wedge \dfrac{d^2\vec{M}}{du^2}\right)^2}{\dfrac{d\vec{M}}{du}, \dfrac{d^2\vec{M}}{du^2}, \dfrac{d^3\vec{M}}{du^3}}$$

or, in rectangular axes,

$$T = \frac{\sum (y'z'' - y''z')^2}{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}$$

We see that the torsion is expressed rationally in terms of the derivatives of the coordinates of a point of the curve C.

207. <u>The intrinsic equations of a space curve</u>. <u>How they determine the curve</u>.

Given a space curve C, along which the vector $\vec{M}(u)$ has continuous derivatives up to the third order, the arc exists together with the curvature and torsion that are functions of it. Henceforth, if we take the arc as the parameter, then R and T are known continuous functions of s. They completely define the form of the curve C, but clearly they can only define it to within a displacement; we say that they define it intrinsically. The equations of the form

$$\frac{1}{R} = \phi(s) \quad , \qquad \frac{1}{T} = \psi(s)$$

are *the intrinsic equations of the curve.*

In order to prove that they define a curve C to within a displacement, we shall show first of all that they define the vectors $\vec{t}, \vec{n}, \vec{b}$ to within a rotation of the coordinate axes. We must then establish the fact that the Frenet-Serret formulae thus determine these vectors when R and T are given as continuous functions of s.

Let us call $\alpha,\beta,\gamma;\ \alpha_1,\beta_1,\gamma_1;\ \alpha_2,\beta_2,\gamma_2,$ the components of $\vec{t}, \vec{n}, \vec{b}$, respectively. Following the formulae in (5), we have

$$\frac{d\alpha}{ds} = \frac{\alpha_1}{R} \quad , \qquad \frac{d\alpha_1}{ds} = \frac{-\alpha}{R} + \frac{\alpha_2}{T} \quad , \qquad \frac{d\alpha_2}{ds} = \frac{-\alpha_1}{T} \tag{8}$$

and the β and γ satisfy this same system. This differential system, which takes the form

$$\frac{dX}{ds} = rY - qZ \ , \quad \frac{dY}{ds} = pZ - rX \ , \quad \frac{dZ}{ds} = qX - pY \ , \qquad (9)$$

with $q = 0$, $r = 1/R$, $p = 1/T$, was discussed in No. 157. Its integration amounts to the integration of a Riccati equation. There is a first integral $X^2 + Y^2 + Z^2$. On the other hand, if X, Y, Z and X_1, Y_1, Z_1 are two systems of solutions, we see that

$$\left(X_1 \frac{dX}{ds} + Y_1 \frac{dY}{ds} + Z_1 \frac{dz}{ds}\right) + \left(X \frac{dX_1}{ds} + Y \frac{dY_1}{ds} + Z \frac{dZ_1}{ds}\right) = 0$$

hence

$$XX_1 + YY_1 + ZZ_1 = \text{const.}$$

It follows that we can find systems of solutions of the system in (8) such that

$$\alpha^2 + \alpha_1^2 + \alpha_2^2 = 1 \ , \qquad \beta^2 + \beta_1^2 + \beta_2^2 = 1 \ , \qquad \gamma^2 + \gamma_1^2 + \gamma_2^2 = 1$$

$$\alpha\beta + \alpha_1\beta_1 + \alpha_2\beta_2 = 0 \ , \quad \beta\gamma + \beta_1\gamma_1 + \beta_2\gamma_2 = 0 \ , \quad \gamma\alpha + \gamma_1\alpha_1 + \gamma_2\alpha_2 = 0$$

It is sufficient for these equations to be satisfied for $s = 0$. Under these conditions, the functions α, β, γ; $\gamma_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ will be the components of three unit vectors forming a trirectangular trihedron; the corresponding vectors $\vec{t}, \vec{n}, \vec{b}$ will be the vectors required. For $s = 0$, we could take arbitrarily the vector $\vec{t}$ (α,β,γ arbitrary with $\alpha^2 + \beta^2 + \gamma^2 = 1$) then $\vec{n}$ such that $\vec{n} \cdot \vec{t} = 0$ and $\vec{b}$ will be determined. *The trihedron is then defined to within a rotation. The vectors* $\vec{t}, \vec{n}, \vec{b}$ *will be completely determined in a unique way, by the formulae of Frenet-Serret, when their positions with respect to the coordinate trihedron for a particular value of* s, *are given.*

In order to determine the curve C when t is known, it suffices to integrate the equation

$$\frac{\overrightarrow{dM}}{ds} = \overrightarrow{t(s)}$$

which is equivalent to

$$\frac{dx}{ds} = \alpha(s) \ , \quad \frac{dy}{ds} = \beta(s) \ , \quad \frac{dz}{ds} = \gamma(s) \quad .$$

We have to make three integrations, which introduce three arbitrary constants corresponding to a translation of the curve C. *The curve* C *is defined to within a displacement (translation and rotation) when the curvature and the*

torsion are given as functions of the arc, provided these functions are continuous.

208. A kinematic interpretation.

The trihedron Mtnb possesses a motion of rotation and a motion of translation with respect to the coordinate axes. If we assume that the arc s represents time, then the speed of the translation is the derivative of M with respect to s. If $\vec{\Omega}$ is the instantaneous rotation vector, then the speed of a point P attached to the Frenet-Serret trihedron, with respect to the coordinate axes, is

$$\vec{t} + \vec{\Omega} \wedge \overrightarrow{MP} \quad .$$

Let us consider the speed of P in the motion of Mtnb with respect to axes Mxyz taken parallel, through M, to the coordinate axes. This speed will be

$$\vec{\Omega} \wedge \overrightarrow{MP}$$

Let us apply this result to the extremities μ, λ, ν of the unit vectors of the axes Mtnb having M as origin. The speeds of these points are the derivatives of t, n, b with respect to s. We therefore have

$$\vec{\Omega} \wedge \vec{t} = \frac{d\vec{t}}{ds} = \frac{\vec{n}}{R}$$

$$\vec{\Omega} \wedge \vec{n} = \frac{d\vec{n}}{ds} = -\frac{\vec{t}}{R} + \frac{\vec{b}}{T}$$

$$\vec{\Omega} \wedge \vec{b} = \frac{d\vec{b}}{ds} = -\frac{\vec{n}}{T}$$

The first relationship shows that $\vec{\Omega}$ is orthogonal to $\vec{n}$; we have

$$\vec{\Omega} = p\vec{t} + r\vec{b}$$

where p and r are scalars. The first and last relationship shows that $r = 1/R$, $p = 1/T$, and the second relationship is then satisfied. *The components* p, q, r *of the vector* $\vec{\Omega}$ *on* Mtnb *are*

$$p = \frac{1}{T} \;,\quad q = 0 \;,\quad r = \frac{1}{R}$$

The determination of the Frenet-Serret trihedron previously described, commencing from the curvature and torsion, is thus seen to be a particular case of the determination of a motion of rotation about a point, when the instantaneous rotation is known. This more general problem reduces to the integration of the system in (9) in the general case, as was the case in No. 157.

Remark. The axis of the helicoidal tangential motion in the displacement of Mtnb with respect to the coordinate axes, is a line parallel to Ω that is obtained by expressing that the speed of one of its points is parallel to $\vec{\Omega}$. We must then have

$$\vec{t} + (p\vec{t} + r\vec{b}) \wedge \vec{MP} = \rho(p\vec{t} + r\vec{b})$$

where P can be taken in the osculating plane; hence $\overrightarrow{MP} = m\vec{t} + m'\vec{n}$.

We see that $m = 0$, and that m' is given by

$$1 - rm' = \rho p \quad , \qquad pm' = \rho r$$

We obtain

$$m' = \frac{r}{r^2 + p^2}$$

209. The sign of the torsion.

The sign of the torsion is tied to the orientation of the space defined by the choice of the coordinate axes. If, without changing Ox, Oy we replace Oz by the opposite direction, then likewise, the vector $\vec{b}$ is replaced by the opposite vector $-\vec{b}$ in the Frenet-Serret trihedron. The Frenet-Serret formulae remain true, while showing that T is changed into $-T$.

But having chosen the axes $Oxyz$, T is defined in magnitude and in sign. Its sign does not depend on the chosen direction of the curve, for, if this direction is changed, then $\vec{t}$, $\vec{b}$ and ds are at once changed to $-\vec{t}$, $-\vec{b}$, and $-ds$, respectively, and the Frenet formulae show that T retains its sign. This can also be seen in the expression for T in terms of the coordinate derivatives.

When T *is positive*, the curve is on the same side of the osculating plane as the binormal when this plane is traversed. We have seen that with respect to this plane, it is

$$Z = \frac{\Delta s^3}{6} \frac{1}{RT} + o(\Delta s^3) \quad .$$

As we assume the coordinate axes to be *dextrorsal* (Mn and Mb are in the plane of the sheet and Mt is above this sheet, assumed to be horizontal (Fig. 71)), observer AB placed on the osculating plane, with his feer on the principal normal at A, his head at B, and looking toward M, will see the curve leaving the osculating plane along his left side. The curve is said to be *dextrorsal*.

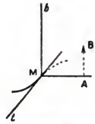

Fig. 71.

When T *is negative*, the curve crosses the plane (for s increasing) on passing from the side opposite to the binormal; the curve is *sinistrorsal.*

Two skew curves that possess the same curvature at each point M, correspond to the same arc value, but the opposed torsions are not equal; *each is equal to a symmetry of the other.*

In effect, the Frenet formulae relative to these two curves are the same when one relates one of the curves to the dextrorsal axes and the other to the sinistrorsal axes. They define curves that are given by the same formulae in these two systems of axes, if we take as the origin of the curves, the origin of the coordinates, for Ox positive, the positive tangent at this point, and for Oy the principal normal. These curves are symmetric to each other with respect to a plane.

210. Particular cases.

The plane curves have zero torsion. The osculating plane is essentially the plane of the curve, the birnormal has a fixed direction, hence the derivative of $\vec{b}$ is zero, and the torsion formula of Frenet-Serret shows that 1/T is zero. Conversely:

Every curve whose torsion is zero at every point, is planar. For the torsion formula (a) shows that the derivative of $\vec{b}$ is zero, hence $\vec{b}$ is a constant vector: $\vec{b} = \vec{b}_0$. The curve is planar. In fact, we have

$$\vec{b}_0\vec{t} = 0\ , \quad \vec{b}_0 \overrightarrow{\frac{dM}{ds}} = 0\ , \quad \vec{b}_0 \cdot (\vec{M} - \vec{M}_0) = 0$$

where M is a point of the curve. The vector $\overrightarrow{M_0M}$ is orthogonal to $\vec{b}_0$; M is in the plane passing through M_0 and orthogonal to $\vec{b}_0$.

The same mode of argument proves that: *A curve whose curvature is constantly zero is a line.* For $\vec{t}$ has a zero derivative, hence has a constant value $\vec{t}_0$; $\vec{M} - \vec{t}_0 s$ has a zero derivative, hence is a constant vector $\vec{M}_0$. We have

$$\vec{M} = \vec{M}_0 + \vec{t}_0 s\ .$$

The converse is straightforward.

CURVES DEFINED BY A PROPERTY OF THE OSCULATING PLANE

If the osculating plane remains parallel to a fixed plane, then the vector b has a fixed direction and following what was said before, *the curve is planar.*

If the osculating plane passes through a fixed point, which can be taken to be 0, then we have

$$\vec{M}\cdot\vec{b} = 0$$

hence, on differentiating with respect to s,

$$\vec{t}\cdot\vec{b} - \vec{M}\frac{\vec{n}}{T} = 0$$

which reduces to

$$\frac{1}{T}\vec{M}\cdot\vec{n} = 0$$

If the forsion is zero, the curve is planar and its plane passes through 0. If the torsion is nonzero, we at once have $\vec{Mn} = 0$ and $\vec{Mb} = 0$, hence $\vec{M}$ and $\vec{t}$ are collinear. But on idfferentiating $\vec{M}\cdot\vec{n} = 0$, we also obtain on account of the second relationship

$$\frac{1}{R}\vec{Mt} = 0$$

Thus, either $\vec{Mt} = 0$, which is incompatible with the fact that $\vec{M}$ and $\vec{t}$ are collinear, or $1/R = 0$, which gives a line. But in this case, the torsion is zero. Hence, *when the osculating plane passes through a fixed point* 0, *the curve is planar and its plane passes through* 0, *and this could be a line passing through* 0.

CURVES DEFINED BY A PROPERTY OF THE TANGENT

The curve whose tangents pass through a fixed point are lines. Since on taking the fixed point at the origin, we have $\vec{M} = \lambda(s)\vec{t}$ and, on differentiating with respect to s,

$$\vec{t}(1-\lambda') = \frac{\vec{n}}{R}$$

This can only occur if $\lambda' = 1$, $1/R = 0$, whence it follows that the curve is a line passing through 0.

The curves whose tangents intersect a fixed line Δ *are plane curves situated in the planes passing through* Δ (or lines intersecting Δ).

For the projection of such a curve C on a plane P perpendicular to Δ is a plane curve whose tangent passes through the foot 0 of Δ in P; it is a line. C is in the plane passing through Δ and this line.

The curves whose tangents at M *are constantly perpendicular to the radius vector* OM, *are spherical curves traced on the spheres of center* 0. For we have

$$\vec{M} \cdot \frac{d\vec{M}}{ds} = 0 , \qquad \vec{M}^2 = \text{const.}$$

REMARK. As an exercise, we could prove these various properties in terms of coordinates.

211. Helices.

A helix is a curve, not reduced to a line, *whose tangent makes a constant angle with a fixed direction* Δ. Let us take an origin 0 on Δ, Δ for the axes 0z, and 0xyz trivectangular. For k the unit vector on 0z, we have by hypothesis

$$\vec{t}\vec{k} = \cos V , \qquad \vec{t} = \frac{d\vec{M}}{ds} \tag{10}$$

where V is fixed. On integrating, we obtain

$$\begin{aligned} z &= \vec{M}\vec{k} \\ &= s \cos V \end{aligned} \tag{11}$$

where we assume, as we may, that the origin of the arcs is taken to be the point of the curve situated in the plane 0xy. Conversely, on differentiating (11), we recover (10). *The property in* (11) *also characterises a helix.*

Let us denote by m the projection of M onto the plane 0xy. The point m describes a curve Γ that is the projection of C; we shall call σ the arc of Γ computed from the same origin as the arc s of C and $\vec{\tau}$ the unit vector of the tangent to Γ at the point m. The projection of $\vec{t}$ onto the direction of $\vec{\tau}$ is $\vec{\tau} \sin V$ (Fig. 72); on 0z it is $\vec{k} \cos V$. We have

$$\vec{t} = \vec{\tau} \sin V + \vec{k} \cos V$$

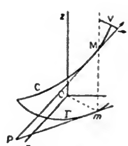

Fig. 72.

On the other hand, since

$$\vec{M} = \vec{m} + z\vec{k}$$

we have, on differentiating with respect to s and taking account of (11),

$$\vec{t} = \frac{d\vec{m}}{ds} + \vec{k} \cos V$$

$$= \vec{\tau} \frac{d\sigma}{ds} + \vec{k} \cos V$$

Comparing these relationships shows that

$$d\sigma = \sin V \, ds \, , \qquad \sigma = \sin Vs \tag{12}$$

We can also define the helix by taking

$$z = \cos Vs$$

$$= \operatorname{cotg} V\sigma$$

z *is proportional to the curvilinear abscissa of* m *on* Γ. *This property is equally characteristic of the helix.* For it implies

$$\vec{M} = \vec{m} + \operatorname{cotg} V\sigma\vec{k}$$

hence

$$\frac{d\vec{M}}{d} = \vec{\tau} + \operatorname{cotg} V\vec{k}$$

$$= \vec{t} \frac{ds}{d\sigma}$$

which in turn, implies, since $\vec{\tau}\vec{k} = 0$,

$$\left(\frac{ds}{d\sigma}\right)^2 = 1 + \operatorname{cotg}^2 V$$

$$= \frac{1}{\sin^2 V} \, ,$$

$$\frac{ds}{d\sigma} = \frac{1}{\sin V}$$

and finally

$$\vec{t}\vec{k} = \frac{d\sigma}{ds} (\vec{\tau} + \operatorname{cotg} V\vec{k})\vec{k}$$

$$= \cos V$$

To the elementary properties thus demonstrated, we can include the fact that the tangent at M intersects the tangent at m at the point P such that the distance Pm is equal to σ.

THE PRINCUPAL NORMAL

By differentiating the relationship (10) with respect to s, we obtain

$$\frac{\vec{n}}{R}\vec{k} = 0 \tag{13}$$

and since we have excluded the lines, we have $\vec{n}\vec{k} = 0$. *The principal normal is perpendicular to* Δ*. Conversely, every nonplanar curve whose principal normal is constantly perpendicular to a fixed line* Δ*, is a helix*, since from $\vec{n}\vec{k} = 0$ we deduce (13) and hence (10).

Moreoverm on differentiating the relationship

$$\vec{t} = \vec{\tau}\sin V + \vec{k}\cos V$$

we have

$$\frac{\vec{n}}{R} = \frac{n_1}{\rho}\sin V\frac{d\sigma}{ds}$$

$$= \frac{n_1}{\rho}\sin^2 V$$

where ρ is the radius of curvature of Γ at m and $\vec{n}_1$ the unit vector of its principal normal. It follows that

$$\vec{n} = \vec{n}_1 , \qquad \rho = R\sin^2 V$$

The relationship between ρ *and* R *is nor characteristic of the helix.*

TORSION. THE BINORMAL

By differentiating the equation $\vec{n}\vec{k} = 0$, we obtain

$$\left(\frac{\vec{b}}{T} - \frac{\vec{t}}{R}\right)\vec{k} = 0 \tag{14}$$

since $\vec{n}\vec{k} = 0$ and $\vec{b}, \vec{t}, \vec{k}$ are coplanar, we have $\vec{b} = w\vec{t} + w'\vec{k}$ and we deduce that $\vec{b}\vec{t} = 0 = w + w'\cos V$ and since $\vec{b}$ and $\vec{t}$ are orthogonal, $\vec{b}, \vec{k}, \vec{t}$ unitary, $w'^2 = 1 + w^2$, showing that

$$\vec{b} = \frac{\varepsilon}{\sin V}(\vec{k} - \cos V\vec{t}) , \qquad \varepsilon = \pm 1$$

which is a geometric relationship apparent in Figure 72. It follows that

$$\vec{b}\vec{k} = \frac{\varepsilon}{\sin V}(1-\cos^2 V)$$

$$= \varepsilon \sin V \tag{15}$$

and (14) is then written as

$$\frac{\varepsilon \sin V}{T} - \frac{\cos V}{R} = 0$$

or

$$T = \varepsilon R \operatorname{tg} V \tag{16}$$

CONVERSE I

If we have constantly on a skew curve $\vec{b}\vec{k}$ = const., *where* $\vec{k}$ *is a fixed vector, then the curve is a helix.* For, on differentiating, we have

$$\frac{\vec{n}}{T}\cdot\vec{k} = 0$$

The torsion is nonzero, since the curve is nonplanar, hence $\vec{n}\vec{k} = 0$, and this implies that we have a helix, as seen above.

CONVERSE II (Barre de Saint-Venant and Bertrand)

A nonplanar curve, along which the ratio of the torsion to the curvature is constantm is a helix.

For, if $R = kT$, where k is a constant, then we have, following the formulae of Frenet-Serret,

$$d\vec{b} + k d\vec{t} = 0, \qquad \vec{b} + k\vec{t} = \vec{k}'$$

where $\vec{k}'$ is nonzero, since $\vec{b}$ and $\vec{t}$ are orthogonal, and on multiplying by $\vec{n}$, we obtain $\vec{k}'\vec{n} = 0$, and the curve is a helix.

A PARTICULAR CASE

If the torsion and the curvature are constant without being zero, then the curve is a circular helix (Puiseaux).

In effect, the curve is a helix on account of the above proposition. If we project it onto the plane perpendicular to the direction Δ with which the tangents form a constant angle, then we obtain a curve Γ whose radius of curvature $\rho = R\sin^2 V$, is constant. It is therefore a circle.

REMARK. The preceding conclusion, namely Γ is a circle by virtue of the fact that it is a plane curve with constant radius of curvature, as proved directly, also results from the fact that a curve is completely determined by its intrinsic

equations which here are $1/T = 0$, $R = \text{const.}$ As a circle of radius R satisfies these conditions, it is the curve required. Similarly, Puiseaux's theorem given above, arises directly from this same proposition.

II. THE METHOD OF THE MOVING TRIHEDRON

This method, as applied by Ribaucour (1871) and Darboux, consists of assigning the elements in question to the axes tied to the Frenet-Serret trihedron.

212. Planar envelopes related to the curve. The normal plane.

In order for a point P to belong to the normal plane at M, it is necessary and sufficient to have $\vec{t}\cdot\overrightarrow{MP} = 0$. The envelope of the normal plane is obtained by differentiating the equation

$$\vec{t}(\vec{P}-\vec{M}) = 0 \tag{17}$$

with respect to s, giving

$$\frac{\vec{n}}{R}(\vec{P}-\vec{M}) - \vec{t}\cdot\frac{d\vec{M}}{ds} = 0$$

or

$$\vec{n}(\vec{P}-\vec{M}) = r \tag{18}$$

The characteristic of the normal plane, which is called *the polar line* or *the axis of curvature*, is therefore, on account of (18), the perpendicular to the osculating plane that cuts the principal normal at the center of curvature. The point of contact of the polar line with its envelope, i.e., the point of the edge of regression of the developable surface that it engenders, situated on this line, is obtained by adding to equations (17) and (18) the derivative of (18) with respect to the parameter s. This derivative is

$$\left(-\frac{\vec{t}}{R}+\frac{\vec{b}}{T}\right)(\vec{P}-\vec{M}) - \vec{n}\vec{t} = \frac{dR}{ds}$$

taking account of (17) it reduces to

$$(\vec{P}-\vec{M})\vec{b} = T\cdot\frac{dR}{ds} \tag{19}$$

The point of contact of the polar line with its envelope is therefore defined, on account of (17), (18), and (19), by

$$\vec{P} = \vec{M} + \vec{n}\cdot R + \vec{b}\cdot T\,\frac{dR}{ds} \tag{20}$$

We see that this is the center of the osculating sphere at M, to the curve

(nos. 57,205). The developable surface engendered by the polar line is called *the polar surface.*

REMARK. When the center P of the osculating sphere is fixed, the normal plane passes through this fixed point P, the tangent is perpendicular to the vector $\overrightarrow{PM}$ and the curve is spherical (no. 210). It is traced on the osculating sphere whose radius is fixed. For this to be the case, it is necessary and sufficient that the vector (20) should be fixed, hence having a zero derivative. Thus we obtain the necessary and sufficient condition

$$R + T\frac{d}{ds}\left(T\frac{dR}{ds}\right) = 0$$

for a curve to be spherical. If $R' \neq 0$, this condition is equivalent to

$$R^2 + T^2\left(\frac{dR}{ds}\right)^2 = \text{const.}$$

which states that the radius of the osculating sphere is constant.

THE ENVELOPE OF THE RECTIFYING PLANE

On differentiating the equation of the plane,

$$\vec{n}(\vec{P} - \vec{M}) = 0$$

with respect to the parameter s, where P is a varying point, we obtain

$$\left(\frac{\vec{b}}{T} - \frac{\vec{t}}{R}\right)(\vec{P} - \vec{M}) = 0$$

The line defined by these equations, known as the rectifying line, can be expressed in the form

$$\vec{P} = \vec{M} + \lambda\vec{t} + \mu\vec{b}$$

where λ and μ are determined by the condition

$$\left(\frac{\vec{b}}{T} - \frac{\vec{t}}{R}\right)(\lambda\vec{t} + \mu\vec{b}) = 0$$

We have $\mu = \omega T$, $\lambda = \omega \dot{R}$, where ω is an arbitrary constant, the rectifying line is parallel to the instantaneous rotation vector $\vec{\Omega}$ of the Frenet-Serret trihedron (no. 208).

The locus of the rectifying line, the envelope of the rectifying plane, is called *the rectifying surface.* To find the point of the line of striction of the rectifying surface corresponding to M, let us take the point P of the rectifying line

$$\vec{P} = \vec{M} + \omega R\vec{t} + \omega T\vec{b}$$

and determine ω in order that the curve so defined is parallel to the rectifying line. We have

$$\frac{d\vec{P}}{ds} = \vec{t}(1+\omega' R+R'\omega) + \vec{b}(\omega' T+\omega T')$$

whence we deduce the condition

$$\frac{1}{\omega R} = \frac{T'}{T} - \frac{R'}{R}$$

The point P is at infinity if ω is infinite, hence if

$$\frac{T'}{T} = \frac{R'}{R}$$

where $R/T = \text{const.}$ Under these conditions, the rectifying line engenders a cylinder, and makes a constant angle with the tangent. We find that the curves whose ratio of curvature to torsion is constant are helices.

213. Developments.

We seek to define the normals to the curve C admitting an envelope. A normal N is determined by the angle θ, which it makes with the principal normal, where θ is measured in the plane Mnb in the usual positive sense, from Mn to Mb. A point P of this normal is defined by

$$\vec{P} = \vec{M} + \rho(\cos\theta\vec{n} + \sin\theta\vec{b})$$

where ρ is the magnitude of the vector $\overrightarrow{MP}$ on the positive direction of the normal N, defined by the vector $\cos\theta\vec{n} + \sin\theta\vec{b}$. We seek to determine ρ and θ as functions of s for which the locus of $\vec{P}$ is tangent to the normal N. We have

$$\frac{\overrightarrow{dP}}{ds} = \vec{t} + \frac{d\rho}{ds}(\cos\theta\vec{n}+\sin\theta\vec{b}) + \rho(-\vec{n}\sin\theta+\vec{b}\cos\theta)\frac{d\theta}{ds}$$

$$+\rho\left(\cos\theta\left(\frac{\vec{b}}{T}-\frac{\vec{t}}{R}\right)-\sin\theta\frac{\vec{n}}{T}\right)$$

$$= \frac{d\rho}{ds}(\cos\theta\vec{n}+\sin\theta\vec{b}) + \left(1-\rho\frac{\cos\theta}{R}\right)\vec{t}$$

$$-\rho\left(\frac{1}{T}+\frac{d\theta}{ds}\right)\sin\theta\vec{n} + \rho\left(\frac{1}{T}+\frac{d\theta}{ds}\right)\cos\theta\vec{b}$$

In order for this vector and the vector $\vec{n}\cos\theta + \vec{b}\sin\theta$ to be collinear, it is necessary and sufficient that the coefficient of $\vec{t}$ is zero; hence

$$\rho\cos\theta = R \tag{21}$$

and that the vectors

$$\rho\left(\frac{1}{T}+\frac{d\theta}{ds}\right)(\cos\theta\vec{b}-\sin\theta\vec{n})$$

$$\vec{n}\cos\theta+\vec{b}\sin\theta$$

are collinear. This last condition can only be realized when

$$\frac{d\theta}{ds}+\frac{1}{T} = 0 \tag{22}$$

We then obtain the solutions by taking

$$\theta = -\int_{s_0}^{s}\frac{ds}{T}+\text{const.}$$

The surfaces engendered by the normals so defined are the developable surfaces containing the curve C; any two of these surfaces intersect at a constant angle along C (Joachimstal's theorem). The edge of regression of any one of these surfaces is defined by the locus of P, where ρ is given by the relationship in (21). The point of contact of Γ and of the normal N, is therefore on the polar line. *All the curves* Γ, *enveloping the normals of* C, *which we call the developments of* C *are situated on the polar surface.*

A PARTICULAR CASE

In order for the principal normal to have an envelope, it is necessary and sufficient to have Equation (22) satisfied for $\theta = 0$, hence the torsion is to be zero and the curve C is planar. In this case, the development enveloping the principal normals is the development considered in planar geometry, situated in the plane of the curve. The other developments are helices traced on the polar surface which, in this case, is a cylinder.

REMARK. The only other cases where a line other than Mt passing through M, and invariably tied to the Frenet-Serret trihedron, has an envelope, are: 1) The case where C is a plane curve (the envelopes are then developments, and the envelopes of lines situated in the plane of the curve C and cutting C at a fixed angle). 2) The case where C is a helix (the envelopes are then helices).

214. Involutes.

An *involute* of a curve C is a curve Γ of which C is a development. We obtain an involute by taking a point P on the tangent at an arbitrary point M of C and showing that P describes a normal curve to MP. We have

$$\vec{P} = \vec{M}+\lambda\vec{t}$$

where λ is a function of s that is to be determined. It follows that

$$\frac{d\vec{P}}{ds} = \vec{t}(1+\lambda') + \lambda\frac{\vec{n}}{R} \tag{22}$$

This vector must be parallel to the normal plane, hence $\lambda' = -1$, $\lambda = -s+s_0$. To each value of s_0 there corresponds an involute of which the varying point P coincides with M when $s = s_0$. This common point M_0 is a point of striction for the involute Γ, since on account of (23), $d\vec{P}/ds$ is zero at this point. We shall take this point as the origin and assume, to fix ideas, that s is positive. If M_1 is the point of C corresponding to $s = s_1 > 0$ and if $s < s_1$, we will have (Fig. 73)

$$|\overrightarrow{MP}| + \text{arc } MM_1 = s_1$$

which gives a mechanical definition for Γ: we assume that a thread held on Γ and fixed at M_1 is spread out, while remaining tangential to Γ and we take the locus of its extremity P.

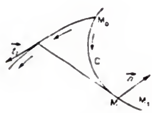

Fig. 73

Let us take on Γ the positive sense defined by the displacement of P when M describes C in the sense of s increasing. Let s_1, $\vec{t}_1$, $\vec{n}_1$, $\vec{b}_1$ be the arc M_0P of Γ and the vectors defining the Frenet-Serret trihedron at P.

By the hypotheses established, the equation of the curve Γ is

$$\vec{P} = \vec{M} = s\vec{t}$$

It follows that

$$\vec{t}_1 = \frac{d\vec{P}}{ds_1}$$

$$= \frac{d\vec{P}}{ds}\frac{ds}{ds_1}$$

$$= -s\frac{\vec{n}}{R}\frac{ds}{ds_1}$$

which yields

$$\vec{t}_1 = -\vec{n}, \qquad s\frac{ds}{ds_1} = R$$

$$s_1 = \int_0^s \frac{s\,ds}{R}$$

Taking R_1 to be the radius of curvature of Γ, we then have

$$\frac{d\vec{t}_1}{ds_1} = \frac{\vec{n}_1}{R_1}$$

$$= -\frac{d\vec{n}}{ds_1}$$

$$= -\frac{ds}{ds_1}\left(-\frac{\vec{t}}{R} + \frac{\vec{b}}{T}\right)$$

$$= \frac{R}{s}\left(\frac{\vec{t}}{R} - \frac{\vec{b}}{T}\right)$$

It follows that

$$\frac{1}{R_1^2} = \frac{R^2}{s^2}\left(\frac{1}{T^2} + \frac{1}{R^2}\right)$$

REMARKS. I. All of the involutes of C are on the developable surface engendered by the tangents of the curve C; these are the orthogonal trajectories of the generators of this surface.

II. In order for Γ to be a plane curve, it is necessary for C to be the development of a plane curve, hence to be a plane curve or a helix. This is sufficient, since if C is a helix, the vector $\vec{n}$ is parallel to a plane, hence $\vec{t}_1$ is parallel to a plane; we have $\vec{t}_1\vec{k} = 0$, $\vec{k} = \text{const.}$, hence $(\vec{P} - \vec{M}_0)\vec{k} = 0$.

III. SOME INTERESTING CURVES

215. Bertrand's curves.

These are curves whose principal normal is also the principal normal of another curve. If C is such a curve, M one of its points and P a point of the principal normal at M, then we have

$$\vec{P} = \vec{M} + k\vec{n}$$

and we seek those conditions under which we can define k as a function of s in order for the locus of P to be a curve Γ admitting $\vec{n}$ as the vector of its principal normal. We have

$$\frac{d\vec{P}}{ds} = \hat{t} + k\left(-\frac{\vec{t}}{R} + \frac{\vec{b}}{T}\right) + k'\vec{n}$$

It is necessary for this vector to be normal to $\vec{n}$, hence $k' = 0$ and so k must be a constant. If s_1 is taken to be the arc of Γ, we have, under these conditions

$$\vec{t}_1 = \frac{d\vec{P}}{ds_1}$$

$$= \left(1 - \frac{k}{R}\right)\frac{ds}{ds_1}\vec{t} + k\frac{ds}{ds_1}\frac{\vec{b}}{T}$$

where $\vec{t}_1$ is the unit vector of the tangent to Γ. It is necessary for its derived vector with respect to s (proportional to the derivative with respect to s_1), to be collinear with $\vec{n}$. As $\vec{t}$ and $\vec{b}$ have their derivatives collinear with $\vec{n}$, it is sufficient to state that the derivatives of the factors of $\vec{t}$ and $\vec{b}$ are zero. We then obtain the necessary and sufficient conditions

$$\left(1 - \frac{k}{R}\right)\frac{ds}{ds_1} = \text{const.}, \quad k\frac{ds}{ds_1}\frac{1}{T} = \text{const.}, \quad k = \text{const.}$$

On factoring out the first two conditions, we have

$$k = \text{const.}, \quad 1 - \frac{k}{R} = \frac{k}{T}\cdot k_1, \quad k_1 = \text{const.}$$

These conditions are satisfied for *circular helices, for any* k, since R and T are constants. Besides the circular helices, the condition will be satisfied if there exists a linear relationship with constant coefficients between the curvature and the torsion, of the form

$$\frac{A}{R} + \frac{B}{T} - 1 = 0, \qquad A \neq 0 \tag{24}$$

and the number $k = A$ will be completely determined. When condition (24) is satisfied, we obtain a pair of curves, each of which is a Bertrand curve. The expression for $\vec{t}_1$ shows that the osculating planes of these curves at the points corresponding to M and P, form a constant angle.

PARTICULAR CASES

The relation occurs if $R = \text{const.}$ We then take $B = 0$ and $k = R$. This particular case was considered by Monge.

A REMARK ON TORSION

Following the expression for $\vec{t}_1$, the angle ω of $\hat{t}$ and $\hat{t}_1$ satisfies the condition

$$\sin\omega = k\frac{ds}{ds_1}\frac{1}{T}$$

Let us permute the roles of the two curves; for T_1 taken to be the torsion of Γ, we shall have

$$\pm\sin\omega = k\frac{ds_1}{ds}\frac{1}{T}$$

and on forming the product, we see that TT_1 is constant.

216. Curves with constant curvature and curves with constant torsion.

For a curve with constant curvature, R is constant. The curve is determined by giving the indicatrix of the tangents. If $\overrightarrow{\sigma(u)}$ is the unit vector defining this indicatrix, then we have

$$\vec{t}(s) = \sigma(\vec{u})$$

$$\frac{d\vec{M}}{ds} = t(\vec{s})$$

$$\left|\frac{d\sigma(\vec{u})}{du}\right| = \frac{d\sigma}{du}$$

$$= \frac{1}{R}\frac{ds}{du}$$

hence

$$\vec{M} = \int \overrightarrow{t(s)}\, ds$$

$$= R\int \overrightarrow{\sigma(u)}\ \frac{d\overrightarrow{\sigma(u)}}{du}\quad du \tag{25}$$

EXAMPLE. If we take the expressions

$$\alpha = \cos^2 u\ ,\quad \beta = \cos u \sin u\ ,\quad \gamma = \sin u$$

for the components of $\sigma(u)$, then the coordinates of M will be given by elementary functions in those related to y and z; x will be given by an elliptic integral.

CURVES WITH CONSTANT TORSION

Here we can give a unit vector $\overrightarrow{\tau(u)}$, defining the indicatrix of the binormals. We then have

$$\frac{d\vec{b}}{ds} = \frac{d\overrightarrow{\tau(u)}}{du} \cdot \frac{du}{ds}$$

$$= -\frac{\vec{n}}{T}$$

and

$$\vec{n} = -T\frac{d\overrightarrow{\tau(u)}}{du} \cdot \frac{du}{ds}$$

$$\vec{t} = \vec{n} \wedge \vec{b}$$

$$= -T\frac{du}{ds}\left(\frac{d\overrightarrow{\tau(u)}}{du} \wedge \overrightarrow{\tau(u)}\right)$$

It follows that

$$\vec{M} = +T\int\left(\overrightarrow{\tau(u)} \wedge \frac{d\overrightarrow{\tau(u)}}{du}\right) du \qquad (26)$$

This expression is more straightforward than formula (25). By taking as above

$$\alpha_2 = \cos^2 u$$

$$\beta_2 = \cos u \sin u$$

$$\gamma_2 = \sin u$$

we will obtain a curve with constant torsion in terms of elementary functions. Moreover, it will suffice that the indicatrix of the binormals is unicursal in order to revert to the integration of rational fractions. Here is an example (due to Fabry) of a unicursal curve with constant torsion: we take as the components of the vector $\vec{b}$:

$$\frac{\sqrt{\lambda}\cos\mu u - \sqrt{\mu}\cos\lambda u}{\sqrt{\lambda} + \sqrt{\mu}}$$

$$\frac{\sqrt{\lambda}\sin\mu u + \sqrt{\mu}\sin\lambda u}{\sqrt{\lambda} + \sqrt{\mu}}$$

$$\frac{2\sqrt[4]{\lambda\mu}\cos\frac{\lambda+\mu}{2}u}{\sqrt{\lambda}+\sqrt{\mu}}$$

where λ and μ are integers.

REMARK. By linearly combining the above results (formula (25) and (26) where we take $\overrightarrow{\sigma(u)} \equiv \overrightarrow{\tau(u)}$), we obtain a curve

$$\vec{M} = A\int \sigma(u)\ |\sigma'(u)|du + B\int (\overrightarrow{\sigma(u)} \wedge \overrightarrow{\sigma'(u)})\ du$$

which is a Bertrand curve, for which

$$\frac{A}{R} + \frac{B}{T} = 1 \ .$$

217. <u>Extremals of</u> $\int f(M)ds$

If we take the integral in its parametric form, then the extremals of

$$\int f(M)\ ds = \int f(x,y,z)\sqrt{x'^2+y'^2+z'^2}\ dt$$

are given by the equations of no. 202, which take the form

$$\sqrt{\Delta}\,\frac{\partial f}{\partial x} - \frac{d}{dt}\left(f\,\frac{x'}{\sqrt{\Delta}}\right) = 0$$

$$\sqrt{\Delta}\,\frac{\partial f}{\partial y} - \frac{d}{dt}\left(f\,\frac{y'}{\sqrt{\Delta}}\right) = 0$$

$$\sqrt{\Delta}\,\frac{f}{z} - \frac{d}{dt}\left(f\,\frac{z'}{\sqrt{\Delta}}\right) = 0$$

where

$$\sqrt{\Delta} = \sqrt{x'^2+y'^2+z'^2}$$

$$= \frac{ds}{dt}$$

By multiplying by dt/ds and introducing the direction cosines α, β, γ of the tangent, we can write these as

$$\frac{\partial f}{\partial x} - \frac{d}{ds}\,(f\alpha) = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{ds}(f\beta) = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{ds}(f\gamma) = 0$$

If we denote the gradient of $f(M)$ by $\overrightarrow{V(M)}$, then these equations can be summarized by a single expression, namely

$$\overrightarrow{V(M)} = \frac{d}{ds}(f(M)\vec{t})$$

By making the operation indicated in the second member, we have

$$\overrightarrow{V(M)} = \frac{df(M)}{ds}\vec{t} + \frac{f(M)}{R}\vec{n}$$

We then deduce the intrinsic equations of the extremals

$$\overrightarrow{V(M)} \cdot \vec{n} = \frac{f(M)}{R}, \qquad \overrightarrow{V(M)} \cdot \vec{b} = 0 \tag{27}$$

The second equation shows that *the osculating plane of an extremal is, at each point* M, *orthogonal to the level surface* $f(M) =$ const. *that passes through this point.*

The curvature is then determined by the first equation (27) when $\vec{n}$ is known.

PARTICULAR CASES

When the level surfaces are parallel to a fixed plane, which can be assumed to be identified with $z = 0$, $f(x,y,z)$ only depends on z (since the partial derivatives with respect to x and y will be identically zero). This necessary condition is clearly sufficient. The property of the osculating plane then implies that the vector $\vec{b}$ is parallel to the plane $z = 0$ in such a way that either this vector is fixed and the extremal is planar, or the indicatrix of the tangents is reduced to a point and the extremal is a parallel to Oz. The extremal planes can be obtained by taking them to be situated in the plane $x = 0$. These will be the extremals of the integral

$$\int f(z)\sqrt{1 + \left(\frac{dy}{dz}\right)^2}\, dz$$

and will be given by the equation

$$f(z)\frac{dy}{ds} = \text{const.}$$

They will be determined by a quadrature.

THE CASE WHERE THE LEVEL SURFACES ARE CONCENTRIC SPHERES

By taking the origin to be the center of these spheres, we see as above, by taking spherical coordinates, that $f(M)$ only depends on r, the distance from M to 0. Conversely, this suffices for the level surfaces to be spheres of center 0. The osculating planes of an extremal will pass through 0; the extremals will be plane curves whose plane passes through 0, or lines passing through 0. We will obtain an extremal by assuming that it is in the plane $0xy$ and determining its equation in polar coordinates (r,θ). We express the integral as

$$\int f(r)\sqrt{dr^2 + r^2 d\theta^2} = \int f(r)\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\ dr$$

the extremals will be given by

$$f(r)\,\frac{r^2 d\theta}{ds} = \text{const.} \tag{28}$$

Again, these will be determined by a quadrature.

REMARKS. I. When we introduce the angle V of the vector $\vec{t}$, taken in the sense of θ increasing, with the direction θ, we know that $r d\theta = \sin V\, ds$ and the equation (28) is written as

$$f(r) r \sin V = \text{const.} \tag{29}$$

When we take $f(M)$ to be an index of refraction, then the integral in question gives, to within a factor, the duration of a travelling light ray. The extremals give the duration of the minimum distance and provide the effective trajectories of light rays following Fermat's principle. The case of spherical and concentric level surfaces corresponds to the propagation of light in the neighborhood of the earth. Formula (29) gives Bouguer's law of refraction.

II. The plane curve, whose intrinsic equation is $Rs = a^2$ $(a = \text{const.})$ and whose equations can be taken in the form

$$x = \int_0^s \cos(bs^2)\, ds\,, \qquad y = \int_0^s \sin(bs^2)\, ds$$

$$b = \frac{1}{2a^2}$$

is also encountered in optics. We can easily construct this curve and show that it has no double points.

IV. IMPORTANT NUMBERS RELATING TO ALGEBRAIC CURVES

218. Stationary points.

A skew curve Γ is said to be algebraic when the coordinates x, y, z of one of its points M can be expressed in terms of algebraic functions of a parameter u. We then have

$$P(x,u) = 0, \quad Q(y,u) = 0, \quad R(z,u) = 0 \tag{30}$$

where P, Q, and R are polynomials. When we eliminate u from the first equation and the last two, we obtain the equations of two algebraic cylinders on which the curve is traced. But we know that the curve Γ will not, in general, be the complete intersection of these two cylinders.

In general, by eliminating u from the combinations of equations (30), we obtain algebraic surfaces passing through Γ. The curve Γ can then also be defined as forming a part of the intersection of two algebraic surfaces. In certain cases, we can find a pair of algebraic surfaces of which Γ will be the complete intersection. But there do exist curves Γ which can only be the partial intersection of two algebraic surfaces regardless of their choice. This is the case for the space cubics.

Among the space algebraic curves the most simple are the unicursal curves, for which x, y, z are rational functions of u.

STATIONARY POINTS

A space curve may be regarded as the envelope of its tangents and as the line of regression of the developable surface enveloping its osculating planes. We know (no. 57) that the osculating plane has as its equation

$$Ax + By + Cz + D = 0 \tag{31}$$

where $A = y'z'' - z'y''$ and where B, C have analogous expressions. As for D, it depends on the coordinates of M and their first and second derivatives. The coefficients A, B, C, D are then algebraic functions of u. When given the osculating plane (31), the curve Γ is obtained by adding to equation (31) the equations obtained by differentiating with respect to the parameter u:

$$\begin{aligned} A'x + B'y + C'z + D' &= 0 \\ A''x + B''y + C''z + D'' &= 0 \end{aligned} \tag{32}$$

and then solving with respect to x, y, z. If the values thus obtained also satisfy the relationship

$$A'''x + B'''y + C'''z + D''' = 0 \tag{33}$$

we say that the corresponding point M is *a stationary point* of the curve.

The values of u providing these points are the zeros of the algebraic function obtained by equating to zero the determinant of the system of equations (31), (32), and (33).

REMARK. The points to the *stationary osculating plane* of no. 57 adopt the role of points of inflexion of plane curves. We are going to see that *the stationary points* are, in general, *points of striction.* This property arises out of the following theorem:

Theorem. *In general,* x', y', z' *are simultaneously zero at a stationary point.*

In effect, to obtain x', y', z', we can differentiate equation (31) and equations (32). On taking account of equations (32), we obtain

$$Ax' + By' + Cz' = 0$$

$$A'x' + B'y' + Cz' = 0$$

$$A''x' + B''y' + C''z' + A'''x + B'''y + C'''z + D''' = 0$$

the equations that determine x', y', z' in the general case where the system (31) and (32) permits the calculation of x, y, z. The values obtained are not zero simultaneously if equation (33) does not hold. On the contrary, if (33) is satisfied, x', y', z' must satisfy a system of three linear, homogeneous equations with nonzero determinant, and are consequently zero.

We note that when (33) is not satisfied, x', y', z' cannot all be zero, hence only the stationary points or the points corresponding to the ramification points of the algebraic functions A, B, C, D are susceptible to giving points of striction.

219. Cayley's formulae.

If we denote by Γ the algebraic space curve in question, the tangents to Γ engender a developable algebraic surface Σ of degree r; r is called the *rank* of Γ. The number n of osculating planes that can be taken to Γ at an arbitrary point, is the *class* of Γ. On the other hand, we consider the number α of points where the osculating plane is stationary, the number β of stationary points, and the four numbers that are about to be defined. The number d of the chords of Γ that pass through an arbitrary point, is the number of multiple points (double points, in general) of the perspective of the curve Γ onto a plane when it is projected from an arbitrary point of view. The number g of lines situated in an arbitrary plane through which we can take two osculating planes to Γ. The number h of points of an arbitrary plane through which we can take two tangents to Γ, the number k of planes bitangent to Γ passing through an arbitrary point.

The Cayley formulae give relationships between these eight numbers and the degree of Γ, in the most simple cases. They arise out of the Plücker formulae for plane curves.

PLÜCKER FORMULAE

Consider a points curve of order μ and class ν, having δ ordinary double points, ρ points of striction of the first kind (cusps), τ double tangents and i points of inflexion. We know (no. 15) that

$$\nu = \mu(\mu-1) - 3\rho - 2\delta \tag{34}$$

If we take the reciprocal polar, then the double points give the double tangents and the points of striction, the points of inflexion of the transformation. We also have

$$\mu = \nu(\nu-1) - 3i - 2\tau \tag{35}$$

The points of inflexion are given by the points of the curve common with the Hessian (no. 12). In general, this number of common points is $3\mu(\mu-2)$, but with respect to the origin, we see that a double point reduces this number by six units; a point of striction reduces it by eight units. We then have

$$i = 3\mu(\mu-2) - 6\delta - 8\rho \tag{36}$$

The correlative formula deduced from this is a combination of three others.

CAYLEY'S FORMULAE

If we cut the developable surface Σ by an arbitrary plane π, then we obtain a curve γ of degree r and of class n. The double points of γ are the points of π through which we can take two tangents to Γ. The points of striction are the points of Γ situated in π; their number m is the degree of Γ. Hence, on account of (34), we have

$$n = r(r-1) - 2h - 3m \tag{37}$$

The number of double tangents to γ is clearly equal to g and the number of points of inflexion is the number α of points where the osculating plane to Γ is stationary. On account of (35) and (36) we therefore have

$$r = n(n-1) - 3\alpha - 2g$$

$$\alpha = 3r(r-2) - 6h - 8m \tag{38}$$

This is the first group of Cayley formulae.

Let us now consider the cone having as vertex an arbitrary point S and passing through Γ. Let us intersect this cone with a plane π'; we obtain a curve γ' of degree m. The class of γ' is the number of tangent planes to the cone that can be taken through an arbitrary point. This is the number of tangents to Γ intersecting this line; this is r. The double points of γ' correspond to the double generators of the cone; this is the number d of chords passing through S. The number of double tangents is k; the points of

inflexion arise from the osculating planes passing through S; there are n of them. The points of striction arise from the β stationary points. By applying the formulae (34), (35), (36), we thus obtain the second group of formulae

$$r = m(m-1) - 2d - 3\beta$$

$$m = r(r-1) - 2k - 3n$$

$$n = 3m(m-2) - 6d - 8\beta$$

We can apply these formulae to the cubics, to the biquadratic intersections of two quadrics, and to the skew curves of the fourth degree that are not biquadratic.

220. The finite geometry and direct differential geometry.

The Cayley formulae, as is the case with all of the formulae relating to algebraic curves and surfaces, are only true over the complex numbers. When there is a restriction to the real numbers, then it is clear that the above results lose their simplicity, but the elementary theory of conics indeed shows that interesting results can be obtained by restricting to the ordinary real space. The various topological considerations that were developed in no. 6 show that one may arrive at some interesting ideas when one restricts attention to the relative position of certain geometrical elements by partly making an abstraction of their proper form.

THE FINITE GEOMETRY is the study of the intersections of curves and surfaces when approached entirely from the real point of view. Take, for example, the case of plane curves; we regard these curves as being formed from a finite number of elementary arcs: an elementary arc is an arc possessing a continuous tangent and bounding along with the chord that joins its extremities, a convex domain, it is therefore rectifiable (I, 39). The order of a curve is the maximum number of points in common with an arbitrary line. Similarly, we view the tangent at a point, points of inflexion, the class, etc. All the closed convex curves will be of the second order and of the second class. The advances in finite geometry were coordinated and developed by Juel (1899-1920); questions relating to curvature were treated by Mukhopadraya. On these matters, we refer to the exposition by Montel (*Bulletin des Sciences mathématiques*, 1924).

DIRECT DIFFERENTIAL GEOMETRY, so called by Bouligand, is concerned with the infinitesimal elements of curves and surfaces from a more general point of view and with regard to set theory. Let us consider, for example, a continuous plane curve. This is a set of points M defined by $x = f(t)$, $y = g(t)$, where f and g are continuous for $0 \leq t \leq 1$. In order to study the tangents, we consider at the point $M_0(t_0)$, the numbers

$$\frac{g(t) - g(t_0)}{f(t) - f(t_0)}, \qquad t \neq t_0$$

Every number m that will be the limiting value of these curves when t tends to t_0 will be the angular coefficient of a "semi-tangent". This set is studied at every point; it constitutes the *contingent* (of the semi-tangents) (this terminology is due to Bouligand). The most elementary result to be obtained was stated by Denjoy in 1910: *The set of points* M *at which the contingent of semi-tangents belongs to an angle with vertex* M *and measure less than* π, *is denumerable* (spatially, one considers a cone).

For a study of these questions, we refer to the book by Bouligand: *Introduction à la geometrie infinitesimale directe.*

SEMI-CONTINUITY

In this kind of geometrical study, one may discuss the notion of upper or lower semi-continuity (due to Baire) that was applied by Tonelli in the calculus of variations. We say that a real function $f(M)$ defined on a set E is *upper semi-continuous* at a point M_0 of this set when, to every positive number ε we have a corresponding η such that the inequality $|\overrightarrow{MM_0}| < \eta$ implies $f(M) < f(M_0) + \varepsilon$. Likewise, lower continuity is defined. Ordinary continuity implies upper and lower semi-continuity and vice-versa.

Chapter XIII

THE THEORY OF SURFACES — THE GENERAL THEOREMS

The original results relating to the curvature of normal sections of surfaces, were due to Euler (1760). The theorem of Meusnier relating the study of curvatures to that of normal sections, dates from 1776. In this same epoch, Monge studied the generation of surfaces regarded as envelopes; this was subsequently developed in his work: *Application de l'Analyse à la géométrie.* At the start of the 19-th century, Dupin, who is associated with the study (1822) of certain cycloids which bear his name and with a theorem on triply-orthogonal systems, introduced the notion of conjugate tangents.

In his original work, *Disquisitiones generales circa superficies curvas* (1828), Gauss systematically applied curvilinear coordinates, defined total curvature commencing from the spherical representation, and made a study of geodesics. As regards the problem of geographic charts, he introduced the general notion of the conformal representation of one surface upon another (1825). In 1830, Minding utilized the idea of geodesic curvature [as denominated by Bertrand (1848)]. In 1848, O. Bonnet extended the theorems of Gauss concerning geodesics and introduced the notion of geodesic torsion. Joachimstal's theorem concerning the lines of curvature, dates from 1846. During the second half of the 19-th century, there were significant advances in the general theory of surfaces: these were mainly due to the works of Liouville, Ribaucour, Lie, Beltrami, Bianchi, Codazzi and Darboux.

In this chapter we shall present the more elementary theorems which arise out of the theories of Euler, Meusnier, Dupin and Gauss.

I. CURVATURE AND THE TWO FUNDAMENTAL FORMS

221. Various forms of the equations of a surface

We know that a surface relative to a system of axes, which can in general be taken to be rectangular, can be presented in various ways. An equation

$$z = f(x,y) \qquad (1)$$

defines a continuous region of the surface, if $f(x,y)$ is continuous in a domain Δ. There exists a tangent plane if $f(x,y)$ is differentiable (I, 121). The first partial derivatives of f are called p and q; the second partial derivatives, if they exist, are r,s,t (the notation is that of

Monge). If the surface is given in the form

$$\Phi(x,y,z) = 0 \,, \tag{2}$$

it suffices that Φ and Φ'_z are continuous and that $\Phi'_z(x,y,z) \neq 0$ at a point satisfying (2) in order that, in the neighborhood of this point, we can put the equation in the form (1). We know how to calculate p,q,r,s,t once Φ possesses the properties assuring the existence of these derivatives (I, 121). The tangent plane has as its equation

$$Z - z = p(X - x) + q(Y - y) \,,$$

where X,Y,Z are the varying coordinates. This holds with respect to any axes, but to affirm that the normal has $p,q,-1$ as its parameters, it is necessary to take the axes to be rectangular. The points at which Φ'_x, Φ'_y, Φ'_z are simultaneously zero, are the singular points of the surface defined by (2).

THE PARAMETRIC FORM

The surface can calso be defined by stating the coordinates of one of its points, as functions of two parameters u and v:

$$x = \phi(u,v) \,, \qquad y = \psi(u,v) \,, \qquad z = \chi(u,v) \,, \tag{3}$$

or, which amounts to the same thing, by giving a vector field tied to a fixed point and whose extremity M describes the surface. Let $\overrightarrow{M(u,v)}$ be this vector. As we did in no. 56, we shall denote this vector and its derivatives, when they exist, by the notations $\vec{M}$, $\vec{M}'_u$, $\vec{M}'_v$, $\vec{M}''_{uu}$, $\vec{M}''_{uv}$, $\vec{M}''_{vv}$.

A point of the surface (3) is said to be ordinary (nos. 42,52) when the three functional determinants

$$A = \frac{D(y,z)}{D(u,v)} \,, \qquad B = \frac{D(z,x)}{D(u,v)} \,, \qquad C = \frac{D(x,y)}{D(u,v)} \,, \tag{4}$$

are not simultaneously zero at this point. The tangent plane then exists and is given by

$$A(X - x) + B(Y - y) + C(Z - x) = 0 \,.$$

In rectangular axes, the normal has as its direction cosines

$$\frac{A}{H} \,, \quad \frac{B}{H} \,, \quad \frac{C}{H} \,, \qquad H = \sqrt{A^2+B^2+C^2} \quad . \tag{5}$$

Remark. We can solve two of the equations (3) in u, v, about an ordinary point, and express the surface in this solvable form. About this point, there exists a bijective correspondence between the surface S defined by (3) and a region of the plane of the (u,v). Through a point of this region of S, there passes a unique curve u = const. and a unique curve v = const.

Unless specified to the contrary, we will consider a surface S whose points are all ordinary. This does not imply that S has no singular lines, but it has no singular points such as conic points.

THE NORMAL VECTOR

By saying that the determinants in (4) are simultaneously nonzero amounts to saying that the vector

$$\vec{M}'_u \wedge \vec{M}'_v$$

is nonzero. This vector is normal at the point M to the surface S.

LINE COORDINATES

Through each point of S, there passes a line $u = \text{const.}$, and a line $v = \text{const.}$; these lines are line coordinates. We can regard S as defined by the lines $u = \text{const.}$ based on a particular line $v = v_0$, or conversely by the lines $v = \text{const.}$ based on $u = u_0$.

<u>Examples</u>. I. A surface of revolution about Oz can be defined by

$$x = \phi(u)\cos v\ , \qquad y = \phi(u)\sin v\ , \qquad z = \psi(u)\ . \tag{6}$$

for $u = u_0 = \text{const.}$, we obtain circles parallel to the axis Oz, and for $v = v_0 = \text{const.}$, we have the meridians situated in the planes $y = x\,\mathrm{tg}\,v_0$. The point or the points of the surface situated on Oz will be singular points for the representation, even if they are ordinary points of the surface written in another form. Through these points there passes an infinity of meridian points.

II. <u>The developable surface</u>. Let us consider the surface defined by the tangents to a curve Λ. For a parameter defining a point P of Λ, we can consider the arc of the curve taken from a fixed point; let u be this arc. Taking $\vec{t}$ to be the unit vector of the positive tangent, a point M of the surface can be defined by

$$\vec{M} = \vec{P} + \vec{t}(v-u)\ . \tag{7}$$

The lines $u = \text{const.}$ are the generators and the lines $v = \text{const.}$ are the involutes of Λ (no. 214). For $u = v$, the point M is a singular point of the surface, since it is on the line of striction (no. 62).

III. <u>The ruled surface</u>. This is a surface engendered by a line depending on one parameter. We can denote the locus of one of its points P, by $\overrightarrow{P(u)}$, and a unit vector on the line, $\overrightarrow{\tau(u)}$, and take

$$\vec{M} = \overrightarrow{P(u)} + \overrightarrow{\tau(u)v}\ .$$

IV. <u>Surfaces of the second degree</u>. For those of this kind of surface that are ruled (as is the case of quadrics), we can take the generators of the two systems to be coordinate lines. We will obtain a rational parametric representation. For the non-ruled surfaces, the use of generators would lead

to a non-real representation. We can then proceed, as we shall do in the following case of the ellipsoid, by utilizing the homofocal surfaces. Consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 , \qquad a > b > c .$$

The homofocal surfaces have as their equations

$$k(u) \equiv \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} - 1 = 0 .$$

For $-a^2 < u < -b^2$, we obtain a two-sheeted hyperboloid and for $-b^2 < v < -c^2$, $k(v) = 0$ defines a one-sheeted hyperboloid. When u and v are varied between these limits, we obtain a system of curves on the ellipsoid. The linear equations in x^2, y^2, z^2,

$$k(o) = 0, \qquad k(u) = 0, \qquad k(v) = 0,$$

allow us to calculate these numbers. Observe that we then obtain

$$k(-t) \equiv \frac{x^2}{a^2-t} + \frac{y^2}{b^2-t} + \frac{z^2}{c^2-t}$$

$$\equiv \frac{t(t+u)(t+v)}{(a^2-t)(b^2-t)(c^2-t)} \quad ; \tag{8}$$

hence, if we set

$$g(t) = t(t+u)(t+v), \qquad h(t) = (a^2-t)(b^2-t)(c^2-t) ,$$

we can write

$$x^2 = -\frac{g(a^2)}{h'(a^2)} , \qquad y^2 = -\frac{g(b^2)}{h'(b^2)} , \qquad z^2 = -\frac{g(c^2)}{h'(c^2)} . \tag{9}$$

To a system of values u, v, there corresponds eight points of the ellipsoid. By considering the roots of x^2, y^2, z^2, we will obtain representations tenable for the eighth-parts of the ellipsoid; each of these eighths corresponds to the rectangle of the plane of the u,v in question.

222. The arc element or line element. Questions of angles.

Henceforth, we shall consider (without re-stating it) an ordinary region of the surface S defined parametrically, where the vector $\overrightarrow{M(u,v)}$ admits first and second continuous partial derivatives; it therefore admits a first and a second differential. We shall assume the axes to be rectangular.

$$d\vec{M} = \vec{M}'_u \, du + \vec{M}'_v \, dv ,$$

$$\overrightarrow{d^2M} = \vec{M}''_{uu}\, du^2 + 2\vec{M}''_{uv}\, dudv + \vec{M}''_{vv}\, dv^2 \quad .$$

It will be clear that, in certain cases to be discussed, in particular those of this section, the existence of the second partial derivatives will not be essential.

THE ARC OF A CURVE DESCRIBED ON THE SURFACE S

A curve Γ described on S will be defined by giving u and v as functions of one parameter. We shall assume that these functions admit the necessary number of derivatives in order for the calculations to be meaningful.

The differential of the arc of Γ is the square root of $dx^2 + dy^2 + dz^2$: it is therefore, the modulus of $d\vec{M}$. We then have

$$ds^2 = d\vec{M}^2 = \vec{M}'^2_u\, du^2 + 2\vec{M}'_u\vec{M}'_v\, dudv + \vec{M}'^2_v\, dv^2 \quad . \tag{10}$$

On account of the formulae in (3), we obtain the components of $\vec{M}'_u\vec{M}'_v$:

$$\vec{M}'_u\,, \qquad \frac{\partial\phi}{\partial u}\,, \qquad \frac{\partial\psi}{\partial u}\,, \qquad \frac{\partial\chi}{\partial u}\,,$$

$$\vec{M}'_v\,, \qquad \frac{\partial\phi}{\partial v}\,, \qquad \frac{\partial\psi}{\partial v}\,, \qquad \frac{\partial\chi}{\partial v}\,,$$

hence

$$E = \vec{M}'^2_u = \left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2 + \left(\frac{\partial\chi}{\partial u}\right)^2$$

$$F = \vec{M}'_u\vec{M}'_v = \frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} + \frac{\partial\chi}{\partial u}\frac{\partial\chi}{\partial v}$$

$$G = \vec{M}'^2_v = \left(\frac{\partial\phi}{\partial v}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2 + \left(\frac{\partial\chi}{\partial v}\right)^2 \quad .$$

The ds^2 *of the surface* S, *or the square of the line element, is given by the quadratic form in* du, dv

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad . \tag{11}$$

This is the first fundamental quadratic form. E, F, G are functions of u and v. By replacing, in this expression, du and dv by their values as functions of parameters defining the line Γ, we obtain the differential of the arc of this line.

A PARTICULAR CASE

When the line coordinates are orthogonal, we have

$$F = \vec{M}'_u\vec{M}'_v = 0 \quad .$$

Conversely, if $F = 0$, the system of line coordinates is orthogonal.

Examples. I. For the surface of revolution defined by the formulae in (6), the system of coordinates is orthogonal, $F = 0$. We obtain

$$ds^2 = [\phi'(u)^2 + \psi'(u)^2]du^2 + \phi(u)^2 dv^2 \quad .$$

II. For the developable surface defined by (7), the system $u = \text{const.}$, $v = \text{const.}$ is also orthogonal. We also have

$$\overrightarrow{dM} = \overrightarrow{dP} + \left(\frac{\vec{n}}{R}(v-u) - \vec{t}\right)du + \vec{t}dv$$

$$= \frac{\vec{n}}{R}(v-u)du + \vec{t}dv$$

$$ds^2 = \overrightarrow{dM}^2 = \frac{(v-u)^2}{R^2}du^2 + dv^2 \quad .$$

III. In the case of the ellipsoid defined by the formulae in (9), we have, for example

$$x^2 = -\frac{a^2(a^2+u)(a^2+v)}{h'(a^2)}$$

and we see that

$$\frac{\partial x}{\partial u} = \frac{x}{2(a^2+u)} , \qquad \frac{\partial x}{\partial v} = \frac{x}{2(a^2+v)} \quad .$$

In order to calculate

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$= \sum \frac{x^2}{4(a^2+u)^2}$$

we follow Jacobi and apply the identity in (8); this yields

$$4E = -k'(u) = \frac{g'(-u)h(-u)-h'(-u)g(-u)}{h(-u)^2} = \frac{g'(-u)}{h(-u)} \quad .$$

Consequently, as the system of coordinates is orthogonal, we have

$$ds^2 = \frac{1}{4}\left[\frac{g'(-u)}{h(-u)}du^2 + \frac{g'(-v)}{h(-v)}dv^2\right] \quad .$$

(The condition of orthogonality is expressed by the equation

$$\sum \frac{x^2}{(a^2+u)(a^2+v)} = \frac{1}{v-u}\sum\left(\frac{x^2}{a^2+u} - \frac{x^2}{a^2+v}\right) = 0 \quad .)$$

THE ANGLE BETWEEN TWO CURVES

Let us consider two curves of the surface passing through the point M. Each is defined by giving u and v as functions of a parameter. On the first curve, u and v has the differentials du and dv respectively, the arc element is ds and the differential of $\vec{M}$ is $d\vec{M}$. On the second curve, the differentials of u and v will be called δu and δv respectively, the arc element is δs and the differential of $\vec{M}$ will be $\delta\vec{M}$. The angle V between these two curves at the point M, where each curve is directed in the sense of increasing values of the parameter, is defined as the geometric angle. Its cosine is given by considering the scalar product of $d\vec{M}$ and $\delta\vec{M}$. We have

$$|d\vec{M}||\delta\vec{M}|\cos V = d\vec{M}\delta\vec{M}$$

$$= (\vec{M}'_u du + \vec{M}'_v dv)(\vec{M}'_u \delta u + \vec{M}'_v \delta v)$$

hence

$$\cos V = \frac{d\vec{M}\delta\vec{M}}{ds\delta s} \tag{12}$$

and

$$ds\delta v \cos V = \vec{M}'^2_u du\delta u + \vec{M}'_u\vec{M}'_v(du\delta v + dv\delta u) + \vec{M}'^2_v dv\delta v$$

$$= Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v \quad ,$$

which gives

$$\cos V = \frac{Edu\delta u + F(du\delta v+dv\delta u) + Gdv\delta v}{ds\delta s} \quad . \tag{13}$$

In these formulae, ds and δs are positive numbers,

$$ds = \sqrt{Edu^2+2Fdudv+Gdv^2}\,, \qquad \delta s = \sqrt{E\delta u^2+2F\delta u\delta v+G\delta v^2} \quad .$$

The vector product gives the sine of V. We have

$$ds\delta s \sin V = |d\vec{M}\wedge\delta\vec{M}| = |(\vec{M}'_u du + \vec{M}'_v dv)\wedge(\vec{M}'_u\delta u + \vec{M}'_v\delta v)|$$

$$= |du\delta v - dv\delta u||\vec{M}'_u\wedge\vec{M}'_v| \quad .$$

In particular, the angle Ω of the coordinate curves which correspond to $u = \text{const.}$ (hence $du = 0$ and $v = \text{const.}$, hence $\delta v = 0$), is given by

$$\cos \Omega = \frac{F}{\sqrt{EG}} \quad .$$

Since the vector product $\vec{M}'_u \wedge \vec{M}'_v$ has A, B, C as components, we have

$$|\vec{M}'_u \wedge \vec{M}'_v| = \sqrt{EG} \sin \Omega = \sqrt{EG-F^2} = H \tag{15}$$

and the formula (14) gives

$$\sin V = \frac{H|du\delta v - dv\delta u|}{ds\delta s} \quad . \tag{16}$$

Remarks. I. So far we have not assumed that the surface S is actually a curved surface. It could equally be a plane with respect to any system of curvilinear coordinates.

II. The equality in (15) is that used in calculating the area of a curved surface (I,149).

III. The numerator of $\cos V$ is the polar form of the first fundamental quadratic form.

ORTHOGONALITY

The condition of orthogonality of two curves is obtained by equating the numerator of $\cos V$ to zero.

In order to obtain the orthogonal trajectories of a family of curves on S and depending on one parameter, we need to find the differential equation of these curves and then we can apply the orthogonality relation. If $g(u,v,\lambda) = 0$ is the equation of the family, where λ is the parameter, then we eliminate λ between this relation and $g'_u\delta u + g'_v\delta v = 0$, which yields the differential equation

$$h\left(u,v, \frac{\delta v}{\delta u}\right) = 0 \quad .$$

We replace $\delta v/\delta u$ in this equation by its value taken from $\cos V = 0$, namely

$$\frac{\delta v}{\delta u} = -\frac{Edu + Fdv}{Fdu + Gdv} ,$$

and we obtain the differential equation of the orthogonal trajectories.

THE SPACE RELATED TO AN ARBITRARY SYSTEM OF COORDINATES

Let us assume that the space is related to an arbitrary system of coordinates. The rectangular cartesian coordinates x, y, z will be functions of three parameters u, v, w. We will assume that these functions have

continuous partial derivatives and that the functional determinant

$$\delta = \frac{D(x,y,z)}{D(u,v,w)}$$

is nonzero in the domain in question. The line element of a curve will be given by

$$ds^2 = dx^2 + dy^2 + dz^2 = A_1 du^2 + A_2 dv^2 + A_3 dw^2$$
$$+ 2B_1 dvdw + 2B_2 dwdu + 2B_3 dudv \ ,$$

with

$$A_1 = \sum\left(\frac{\partial x}{\partial u}\right)^2, \qquad A_2 = \sum\left(\frac{\partial x}{\partial v}\right)^2, \qquad A_3 = \sum\left(\frac{\partial x}{\partial w}\right)^2,$$

$$B_1 = \sum \frac{\partial x}{\partial v}\frac{\partial x}{\partial w}, \qquad B_2 = \sum \frac{\partial x}{\partial w}\frac{\partial x}{\partial u}, \qquad B_3 = \sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} \ .$$

(The summation is taken over x, y, z.) By taking the square of δ, we see that

$$\delta^2 = \begin{vmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & A_1 \\ B_2 & B_1 & A_3 \end{vmatrix}$$

[cf. (I,154)]. When the coordinate surfaces $u = \text{const.}$, $v = \text{const.}$, $w = \text{const.}$ are pairwise orthogonal, the B's are zero; we say that the coordinate surfaces form a triply-orthogonal system. This is the case for spherical and polar coordinates of the space. We can also utilize a system of ellipsoids and homofocal hyperboloids: the ds^2 obtained will give for $w = \text{const.}$, the ds^2 obtained for the ellipsoid as above.

223. The fundamental form relative to curvatures. The second quadratic fundamental form. Meusnier's theorem.

Let us consider a curve Γ on the surface S and let us denote by $\vec{t}$, $\vec{n}$ the unit vectors of the tangent and the principal normal respectively at the point M and by R, the radius of curvature of Γ at this point. We know that

$$\frac{\vec{n}}{R} = \frac{d^2\vec{M}}{ds^2} \ , \tag{17}$$

where s is the arc of Γ. Let us denote by $\vec{N}$ the unit vector of the normal to S at the point M, i.e. the vector defined by

$$\vec{N} = \frac{\vec{M}'_u \wedge \vec{M}'_v}{H}$$

[formula (15)]. Let us take a scalar multiplication of (17) by $\vec{N}$. We obtain

$$\frac{\vec{n}\vec{N}}{R} = \frac{d^2\,\vec{M}}{Hds^2}\,(\vec{M}'_u \wedge \vec{M}'_v) \quad .$$

If we denote by λ, the angle of the vectors $\vec{n}$ and $\vec{N}$, we see that this equality may be written as

$$H\,\frac{\cos\,\lambda}{R} = \frac{(\vec{M}'_u,\vec{M}'_v d^2\vec{M})}{ds^2} \quad . \tag{18}$$

The mixed product of the numerator of the second member is a sum of three mixed products and, on expanding, may be expressed as

$$(\vec{M}'_u,\vec{M}'_v,\vec{M}''_{uu})du^2 + 2(\vec{M}'_u,\vec{M}'_v,\vec{M}''_{uv})dudv + (\vec{M}'_u,\vec{M}'_v,\vec{M}''_{vv})dv^2 \quad .$$

We denote by D, D' and D" the mixed products appearing in this expression, and we can write the equality in (18) under the expanded form

$$\sqrt{EG-F^2}\,\frac{\cos\,\lambda}{R} = \frac{Ddu^2 + 2D'dudv + D''dv^2}{Edu^2 + 2Fdudv + Gdv^2} \,. \tag{19}$$

The quadratic form

$$Ddu^2 + D'dudv + D''dv^2$$

is the second fundamental form. D, D' *and* D" *are the determinants formed with the first and second partial derivatives of* x, y, z *with respect to* u *and* v. *We have*

$$D = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x''_{u^2} & y''_{u^2} & z''_{u^2} \end{vmatrix} , \qquad D' = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x''_{uv} & y''_{uv} & z''_{uv} \end{vmatrix} , \qquad D'' = \begin{vmatrix} z'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x''_{v^2} & y''_{v^2} & z''_{v^2} \end{vmatrix} ,$$

where D, D' *and* D" *are the Gaussian determinants.*

We also introduce the functions $\mathfrak{L}$, $\mathfrak{M}$ and $\mathfrak{N}$, defined by

$$\mathfrak{L} = \frac{D}{H}\,, \qquad \mathfrak{M} = \frac{D'}{H}\,, \qquad \mathfrak{N} = \frac{D''}{H} \quad .$$

The formula in (19) will then be written as

$$\frac{\cos\,\lambda}{R} = \frac{\mathfrak{L}du^2 + 2\mathfrak{M}dudv + \mathfrak{N}dv^2}{Edu^2 + 2Fdudv + Gdv^2} \quad . \tag{20}$$

The numerator of the second member is the product $\vec{N}d^2\vec{M}$.

THE CASE WHERE THE SURFACE IS TAKEN TO BE IN THE SOLVABLE FORM

When the equation of the surface is taken to be of the form $z = f(x,y)$, we have, on taking x and y as parameters,

$$ds^2 = dx^2 + dy^2 + dz^2 + (pdx + qdy)^2$$

$$= (1 + p^2)dx^2 + 2pqdxdy + (q^2 + 1)dy^2 \quad .$$

Under these conditions, we have, in the notation of Monge,

$$H^2 = EG - F^2 = 1 + p^2 + q^2, \qquad D = r\ , \qquad D' = s, \qquad D'' = t\ ,$$

$$s = \frac{\partial^2 z}{\partial x \partial y}\ ,$$

and the formula in (19) becomes

$$\sqrt{1+p^2+q^2}\ \frac{\cos\lambda}{R} = \frac{rdx^2 + 2sdxdy + tdy^2}{(1+p^2)dx^2 + 2pqdxdy + (1+p^2)dy^2} \tag{21}$$

CURVED SURFACES

When S or part of S, is a plane, r,s,t are zero. Conversely, if r,s,t are zero in a part of S, then p and q, whose differentials are zero, are constants and the part in question forms part of a plane. Leaving this case aside,the surface S will be a curved surface in all of its parts; we say that it is a curved surface.

The curved surfaces are such that the second fundamental form is non-zero, for any du and dv. The points at which r,s,t are zero, are the *flat* points.

On a curved surface r,s,t can be simultaneously zero on lines. For example, if $z = x^3 f_1(x,y)$, r,s,t (and p and q) are zero for $x = 0$.

<u>The first theorem on curvature</u>. The second member of the formula in (20) only depends on the point $M(u,v)$ and on the direction of the tangent to Γ at this point (i.e. on the ratio dv/du). Likewise for the first member:

$$\frac{\cos\lambda}{R} \tag{22}$$

has the same value for all curves of the surface S *passing through* M *and having the same tangent at this point.*

It follows that at a point M of S at which the second fundamental form is identically zero, all of the curves of S whose osculating plane is not identified with the tangent plane to S at M, have an infinite radius of curvature.

Let us consider a point where $\mathfrak{L}^2 + \mathfrak{M}^2 + \mathfrak{N}^2$ *is nonzero.* Let Γ be a curve passing through this point and for which

$$\mathfrak{L}du^2 + 2\mathfrak{M}dudv + \mathfrak{N}dv^2 \neq 0\ . \tag{23}$$

The radius of curvature is given by the equation (20), which can be solved with respect to R. *We have assumed* $R \neq 0$. All the curves Γ

passing through M and having the same osculating plane, have the same center of curvature. For the expression (22) has a same nonzero value. When $\cos\lambda = 0$, R is zero. Hence $\cos\lambda \neq 0$, where R is positive. The product $\vec{n}\cdot\vec{N}$ has a determined value since $\vec{n}$ is in the osculating plane, hence it is known to within a direction. The numbers $\cos\lambda$ and R are determined; the direction of the principal normal is the same for all these curves and the center of curvature is the same. When the expression in (23) is zero, R is infinite when $\cos\lambda \neq 0$. We then obtain this first result.

I. *All the curves* S *passing through a point* M *and having at this point the same tangent and the same osculating plane, have the same center of curvature when condition (23) is satisfied. When condition (23) is not satisfied by the tangent, then all the curves passing through* M *and whose osculating plane is not tangent to* S, *have a zero curvature.*

Let us remark at once that condition (23) is always satisfied when the second form is defined at the point M, i.e. if

$$\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} < 0.$$

Example. Consider the surface $z = xy$; we have at the origin [formula (21)]

$$\frac{\cos\lambda}{R} = \frac{2dxdy}{dx^2 + dy^2} .$$

The curves of the surface which are projected onto the plane Oxy along the circles $x^2 + y^2 - 2ax = 0$, have as their equations

$$x = 2a\sin^2 t , \qquad y = 2a\sin t\cos t , \qquad z = 4a^2\cos t\sin^3 t .$$

Their tangent at the origin $(t = 0)$ is Oy; the condition (23) is not realized. The osculating plane is the plane Oxy, the radius of curvature is a, and the center of curvature is displaced on Ox. We shall see that this a general fact.

Corollary. *The center of curvature at* M *of a curve* Γ *of* S, *is the same as the center of curvature, at this point, of the section of* S *through the osculating plane to* Γ *at* M, *except when the osculating plane is tangent to* S *and when the tangent satisfies the condition* $\mathfrak{L}du^2 + 2\mathfrak{M}dudv + \mathfrak{N}dv^2 = 0$.

Meusnier's Theorem. *The axes of curvature at* M *of all the curves of* S *tangent at* M *to the same line* MT, *intersect the normal to* S *at* M *at the same point providing* MT *satisfies the condition (23).*

In effect, the value of the expression (22) is a nonzero constant, $1/\rho$ say. We have $R = \rho\cos\lambda$. If we consider the vector $\overrightarrow{MI} = \rho\vec{N}$, the vector $\overrightarrow{MC} = R\vec{n}$ is the projection of $\overrightarrow{MI}$ onto the principal normal $\vec{n}$ (fig. 74). The line IC is seen to be perpendicular to the osculating plane TMC.

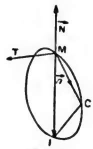

Fig. 74.

We can also say that *the locus of the centers of curvature at* M *of the curves in question, is a circle whose center is on the normal to* S *at* M *which intersects* MT *orthogonally.*

224. The study of curvatures of normal sections. The indicatrix of Dupin.

The above two theorems reduce the study of curvatures to that of normal sections. In order to study these, we may take M to be the origin and assume that the surface is tangent at O to the plane Oxy. By Taylor's formula, its equation is then

$$z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + o(x^2 + y^2) \quad ;$$

p and q are zero at the origin and the vector $\vec{N}$ has for parameters at this point, the values of -p, -q, 1, hence 0,0,1. This is the unit of vector of Oz. Formula (21) here becomes

$$\frac{\cos\lambda}{R} = \frac{rdx^2 + 2sdxdy + tdy^2}{dx^2 + dy^2} \quad .$$

Here, we shall consider the normal sections exclusively. $\cos\lambda$ is equal to +1 or -1; it takes the sign of the second member, since R is positive. *We shall set* $R_N = \pm R$, *by taking,* + *if* $\cos\lambda = +1$, - *if* $\cos\lambda = -1$. *The curvature will then have a sign.* If it is positive the concavity of the section is turned upwards (the sense of Oz positive); if it is negative, the concavity is turned downwards. R_N *is also the direction of the center of curvature of the normal section.* We shall denote by ω, the angle of the tangent with Ox, such that

$$\frac{dx}{\sqrt{dx^2+dy^2}} = \cos\omega \quad , \qquad \frac{dy}{\sqrt{dx^2+dy^2}} = \sin\omega \quad ,$$

and the formula giving the normal curvatures becomes

$$\frac{1}{R_N} = r\cos^2\omega + 2s\cos\omega\sin\omega + t\sin^2\omega \quad , \tag{24}$$

We can simplify this formula by rotating about the axes Oxy. The formula can be written as

$$\frac{1}{R_N} = \frac{r+t}{2} + s \sin 2\omega + \frac{r-t}{2} \cos 2\omega \quad ,$$

and we know that we can only fix a single trignometric line; we will have

$$\frac{1}{R_N} = \frac{r+t}{2} + \sqrt{s^2 + \left(\frac{r-t}{2}\right)^2} \cos(2\omega - 2\theta) \quad ,$$

when we take

$$\cos 2\theta = \frac{r-t}{2\delta} , \qquad \sin 2\theta = \frac{s}{\delta}$$

$$\delta = \sqrt{s^2 + \left(\frac{r-t}{2}\right)^2}$$

setting $\omega = \theta + \omega'$, we then obtain

THE REDUCED FORMULA DUE TO EULER

$$\frac{1}{R_N} = r' \cos^2\omega' + t' \sin^2\omega' \quad , \tag{25}$$

with

$$r' + t' = r + t , \qquad r't' = rt - s^2 \quad .$$

In the form (25), the discussion is immediate. We have three cases tc distinguish.

THE FIRST CASE. We have $rt - s^2 > 0$, hence $r't' > 0$. The second member of (25) always has the same sign and all the normal sections have their concavity directed in the same sense. The surface is on a side determined by the tangent plane [cf. (I,129)], and $\frac{1}{R_N}$ varies from r' to t'. We obtain the same curvature for the sections ω' and $-\omega'$. We say that the point M is *elliptic*.

THE SECOND CASE. We have $rt - s^2 < 0$, hence $r't' < 0$. The curvature is zero for the values ω_0' such that $\operatorname{tg}^2\omega_0' = -\frac{r'}{t'}$. The surface intersects its tangent plane along a curve having a double point at 0 (I,128,129) and the lines $\omega' = \pm\omega_0'$ are the tangents at this point. For $|\omega'| < |\omega_0'|$, R_N takes the sign of r'; when $|\omega'|$ increases from 0 to $|\omega_0'|$, R increases from $1/|r'|$ to infinity. For $|\omega_0| < |\omega'| \leqslant \pi/2$, R_N takes the sign of t' and R increases from $1/|t'|$ to infinity when $|\omega'|$ decreases from $\pi/2$ to $|\omega_0'|$. The curvatures are the same for ω' and $-\omega'$. The point is said to be *hyperbolic*.

THE THIRD CASE. We assume that $rt - s^2 = 0$, hence $r't' = 0$. We can assume that $t' = 0$, $r' \neq 0$. (If $r' = t' = 0$, $r = s = t = 0$, the discussion is no longer purposeful; R_N is infinite for all normal sections.) R_N always takes the sign of r' and R increases from $1/|r'|$ to infinity when $|\omega'|$ increases from 0 to $\pi/2$. There is always the same symmetry. The point is said to be *parabolic*.

ASYMPTOTIC TANGENTS

In the second and third cases there are one or two tangents for which R is infinite. They are known as *asymptotic tangents*. Let us assume, as we may, that Ox is an asymptotic tangent. The equation of the surface is then

$$z = axy + by^2 + o(x^2 + y^2) \ ;$$

a curve Γ of the surface, tangent to this asymptotic tangent, is defined by its projection onto Oxy. If we take the osculating plane to Γ at O, to be Oxy, then this projection has as its equation $y = \alpha x^2 + \ldots$ and we then have $z = a\alpha x^3 + \ldots$; the center of curvature is the point $\frac{1}{2\alpha}$ of Oy, it is arbitrary.

On returning to the general case where the curvatures are given by the formula (20), the above three cases correspond to the three possibilities on the second fundamental quadratic form: *if it is defined,* $\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} < 0$, the normal curvature is never zero, and *the point is elliptic; if it is not defined,* $\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} > 0$, the surface crosses its tangent plane, the asymptotic tangents are defined by

$$\mathfrak{L}du^2 + 2\mathfrak{M}dudv + \mathfrak{N}dv^2 = 0 \quad ,$$

and *the point is hyperbolic. If the form is semi-definite,* $\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} = 0$, there exists a single asymptotic tangent and *the point is parabolic.* The general result which has just been given on the subject of tangent lines to an asymptotic tangent and whose osculating plane is tangent to the surface, justifies the reservation which was made in the Corollary of Theorem I of no. 223.

MEAN CURVATURE AND TOTAL CURVATURE

On account of formulae (24) and (25), *the half-sum of the algebraic curvatures of two normal rectangular sections, is the same for all pairs of such sections.* They correspond in effect to ω and $\omega + \frac{\pi}{2}$; the half-sum of the curvatures is $\frac{r+1}{2}$. We call this quantity the *mean curvature* of the surface at the point M.

The *total curvature* is the number $rt - s^2$. This is the product of the maxima and minima curvatures when $rt - s^2 > 0$ and when $rt - s^2 < 0$, it is the product of the finite maxima and minima of the curvature.

THE INDICATRIX OF DUPIN

This gives a geometric interpretation of the Euler formula in no. (25). Again, let us consider R with its usual meaning, $R > 0$; the formula (25) becomes

$$\varepsilon = r'R\cos^2\omega' \mp t'R\sin^2\omega' \quad ,$$

where ε is equal to $+1$ when ω' gives a positive value to the second member and equal to -1 when the second member is negative. If we consider the point with polar coordinates ρ', ω', then the equation

$$\varepsilon = (r'\cos^2\omega' + t'\sin^2\omega')\rho'^2 \quad ,$$

which is written as

$$\varepsilon = r'X^2 + t'Y^2 \quad ,$$

when we take for the axes of the X the line $\omega' = 0$ and for OY the line $\omega' = \frac{\pi}{2}$, represents:

an ellipse if $r't' > 0$, or $rt - s^2 > 0$, we then take $\varepsilon r' > 0$;

two conjugate hyperbolas if $r't' < 0$, or $rt - s^2 < 0$, one of the hyperbolas corresponds to $\varepsilon = 1$, the other to $\varepsilon = -1$;

two parallel lines equidistant from OY if $r' \neq 0$, $t' = 0$, or $rt - s^2 = 0$, we then take $\varepsilon r' > 0$.

These are the curves that constitute the indicatrix of Dupin. When the indicatrix is known, we obtain the curvature of a normal section by taking $R = \rho'^2$, where ρ' is the distance to 0 of one of the points of intersection of the tangent with the indicatrix.

PRINCIPAL SECTIONS AND PRINCIPAL TANGENTS

The normal sections passing through the axes of the indicatrix are called *principal sections*. The radii of curvature of these sections are the *radii of principal curvature*; the centers of curvature at 0 are the *centers of principal curvature*. The corresponding tangents which are the axes of the indicatrix, are the *principal tangents*. They correspond to the nonzero relative maxima and minima of the curvature, except in the parabolic case, where one of the principal tangents is simultaneously an asymptotic tangent.

In the hyperbolic case, the asymptotes of the Dupin indicatrix are evidently the tangents which were called asymptotic tangents.

When $r' = t'$, all the tangents are principal, the indicatrix being a circle. The point 0 in question is called UMBILIC.

APPLICATION

I. If two surfaces are tangent at a point 0 and intersect along a line Γ passing through 0 and admitting a tangent OT at 0, then the sections of

two surfaces through the normal plane passing through OT have the same center of curvature at the point O. Since, on account of Meusnier's theorem, this center is on the axis of curvature of Γ at the point O. The tangent OT must therefore intersect the indicatrices of the two surfaces at the same point. This proves that the tangents at O to the common curve of the two surfaces belong to the sheaf of lines obtained by joining the origin to the common points of the two indicatrices. But not all the lines of this sheaf are tangents to the line Γ. *It is necessary to take into account, the sign of the algebraic curvatures for which the Dupin indicatrix excludes.* Let us consider the two surfaces

$$z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + o(x^2 + y^2)$$

$$z_1 = \frac{1}{2}(r_1x^2 + 2s_1xy + t_1y^2) + o(x^2 + y^2) \quad ;$$

the projection of the common curve onto Oxy is obtained by stating that $z_1 = z$.

After multiplying by 2, we obtain

$$(r - r_1)x^2 + 2(s - s_1)xy + (t - t_1)y^2 + o(x^2 + y^2) = 0 \quad . \tag{26}$$

In cartesian coordinates x,y, the indicatrices are

$$\varepsilon = rx^2 + 2sy + ty^2 \ , \qquad \varepsilon_1 = r_1x^2 + 2s_1xy + t_1y^2 \ ,$$

where ε and ε_1 are equal to $+1$ or to -1. When ε and ε_1 have the same sign, the central chords common to the conics (27) are obtained by subtracting the equations in (27) and provide the tangents at the origin of the curve (26) when the origin is a double point. But if $\varepsilon_1\varepsilon_2 = -1$, the common central chords will be given by $(r + r_1)x^2 + 2(s + s_1)xy + (t + t_1)y^2 = 0$ and are not tangent to the curve (26).

We must therefore be careful in applying this method. For example, if the two surfaces admit the point O in question as an elliptic or parabolic point and are on the same side of the tangent plane in the neighborhood of O, then the common central chords effectively give the tangents required.

II. If two surfaces are tangential along a line Γ, then their Dupin indicatrices taken at the various points of Γ, are bitangent. When one of the surfaces is a plane, then all of the points of Γ will then be parabolic on the second surface.

Remark. Let S and Σ be two surfaces which have a line Γ in common and let M be a point of Γ at which the normals to S and Σ are distinct. The tangent at M to Γ is the intersection of the tangent planes to S and Σ at M. Let C and C' be the centers of curvature at M of the sections of S and Σ through the normal planes passing through MT. Following

Meusnier's theorem, the osculating plane to Γ at M, is perpendicular to the line CC'.

Remark. If we intersect the surface

$$z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + o(x^2 + y^2)$$

by a plane $z = \frac{\varepsilon}{2}\eta^2$, $\varepsilon = \pm 1$, and if we project the curve so obtained onto the plane $z = \frac{\eta}{2}$, where the center of projection is the origin, then we obtain the curve

$$\varepsilon = rx^2 + 2sxy + ty^2 + o(\eta^2 x^2 + \eta^2 y^2)/\eta^2 \quad .$$

When η tends to zero, we obtain the Dupin indicatrix.

225. The calculation of the radii of principal curvature in the general case

The curvature of a normal section is generally given by the formula in (20), in which $\cos\lambda = \vec{n}.\vec{N} = \pm 1$. If we again take the radius of curvature with a sign, where R_N represents the magnitude of the vector $R\vec{n}$ in the direction $\vec{N}$ (i.e. the abscissa on the normal $\vec{N}$ from the center of curvature of the section in question), then we have

$$\frac{n\vec{N}}{R} = \frac{R\vec{n}.\vec{N}}{R^2} = \frac{R_N}{R_N^2} = \frac{1}{R_N} ,$$

and R_N is given by

$$\frac{1}{R_N} = \frac{\mathcal{L}du^2 + 2\mathcal{M}dudv + \mathcal{N}dv^2}{Edu^2 + 2Fdudv + Gdv^2} .$$

The radii of principal curvature correspond to the finite maxima and minima of R_N, when $\frac{dv}{du}$ is varied. When we set $\frac{dv}{du} = m$, we have to find the maxima and minima of

$$\frac{1}{R_N} = \frac{\mathcal{L} + 2\mathcal{M}m + \mathcal{N}m^2}{E + 2Fm + Gm^2} .$$

These values correspond to the values of m annihilating the derivative: for these values m_1 and m_2, the function of the second member will be equal to the ratio of the derivatives of the numerator and the denominator; we will have

$$\frac{1}{R_N} = \frac{\mathcal{M} + \mathcal{N}m}{F + Gm} = \frac{\mathcal{M}du + \mathcal{N}dv}{Fdu + Gdv} ,$$

$$\frac{dv}{du} = m \quad (m = m_1, m_2) \quad .$$

Similarly, by setting $\frac{du}{dv} = n$, we will obtain

$$\frac{1}{R_N} = \frac{\mathfrak{L}n + \mathfrak{M}}{En + F} = \frac{\mathfrak{L}du + \mathfrak{M}dv}{Edu + Fdv} \quad ,$$

$$\frac{du}{dv} = n = \frac{1}{m} \quad (m = m_1, m_2) \quad .$$

It follows that the values of R_N and the values corresponding values of $\frac{dv}{du}$ are given by the equality

$$\frac{1}{R_N} = \frac{\mathfrak{M}du + \mathfrak{N}dv}{Fdu + Gdv} = \frac{\mathfrak{L}du + \mathfrak{M}dv}{Edu + Fdv} \quad .$$

The required values of R_N will be obtained by eliminating m from the equations

$$m(F - \mathfrak{M}R_N) + E - \mathfrak{L}R_N = 0 \quad ,$$

$$m(G - \mathfrak{N}R_N) + F - \mathfrak{M}R_N = 0 \quad ,$$

$$m = \frac{dv}{du} \quad ,$$

which gives *the equation of the radii of principal curvature*

$$(F - \mathfrak{M}R_N)^2 - (E - \mathfrak{L}R_N)(G - \mathfrak{N}R_N) = 0 \quad . \tag{28}$$

It is effectively of the second degree when $\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} \neq 0$. The mean curvature and the total curvature are the half-sum and the product of the inverse roots of the equation. Denote by R_1 and R_2 the radii of principal curvature; these are positive or negative numbers since they are values of R_N, the roots of (28). We have

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{E\mathfrak{N} - 2F\mathfrak{M} + G\mathfrak{L}}{2(EG - F^2)} \tag{29}$$

$$\frac{1}{R_1 R_2} = \frac{\mathfrak{L}\mathfrak{N} - \mathfrak{M}^2}{EG - F^2} \quad ,$$

or, by introducing Gauss determinants,

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{ED'' - 2FD' + GD}{2(EG-F^2)^{3/2}} \quad , \tag{30}$$

$$\frac{1}{R_1 R_2} = \frac{DD'' - D'^2}{(EG - F^2)^2} \quad .$$

THE PARTICULAR CASE WHERE x AND y ARE PARAMETERS

In this case, we can apply the preceding formulae. But our calculation of the radii of principal curvatures, will be somewhat different. Following (21), we now have

$$\frac{\sqrt{1+p^2+q^2}}{R_N} = \frac{r+2sm+tm^2}{1+p^2+2pqm+(1+q^2)m^2}\,, \qquad m = \frac{dy}{dx}\ .$$

Let us denote the first member of this equality by $\frac{1}{\rho}$. The values of m giving the maximum and minimum of ρ are the values for which the equation in m, obtained by regarding ρ as fixed and m as a variable, has a double root. This equation is

$$\frac{1+p^2}{\rho} - r + 2\left(\frac{pq}{\rho} - s\right)m + \left(\frac{1+q^2}{\rho} - t\right)m^2 = 0\ .$$

The equation in $\frac{1}{\rho}$ is therefore

$$\left(\frac{pq}{\rho} - s\right)^2 - \left(\frac{1+p^2}{\rho} - r\right)\left(\frac{1+q^2}{\rho} - t\right) = 0\ ,$$

or

$$\frac{1+p^2+q^2}{\rho^2} + (2pqs - (1+p^2)t - (1+q^2)r)\,\frac{1}{\rho} + rt - s^2 = 0\ ,$$

$$\frac{1}{\rho} = \frac{\sqrt{1+p^2+q^2}}{R_N}\ .$$

We have

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{(1+p^2)t + (1+q^2)r - 2pqs}{2(1+p^2+q^2)^{3/2}}\ ,$$

$$\frac{1}{R_1R_2} = \frac{rt - s^2}{(1+p^2+q^2)^2}\ .$$

226. The Gauss theorem on total curvature

Gauss showed that the total curvature depends only on E,FG and on their first and second partial derivatives with respect to u and v. In order to establish the theorem, it suffices, following the formulae in (30), to prove that $DD'' - D'^2$ can be expressed in terms of E,F,G and their derivatives. D,D',D'' are the given determinants; let us take the square of D' by multiplying lines by lines. We obtain

$$D'^2 = \begin{vmatrix} E & F & \frac{1}{2}\frac{\partial E}{\partial v} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{1}{2}\frac{\partial E}{\partial v} & \frac{1}{2}\frac{\partial G}{\partial u} & \sum x''^2_{uv} \end{vmatrix}\ .$$

Likewise, let us form the product DD'', line by line. We obtain

$$DD'' = \begin{vmatrix} E & F & \sum x'_u x''_{v^2} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial v} \\ \frac{1}{2}\frac{\partial E}{\partial u} & \sum x'_v x''_{u^2} & \sum x''_{u^2} x''_{v^2} \end{vmatrix} .$$

In order to prove the theorem, we see that it suffices to show that the three expressions

$$\sum x'_u x''_{v^2} , \qquad \sum x'_v x''_{u^2} , \qquad \sum x''^2_{uv} - \sum x''_{u^2} x''_{v^2} = \tau ,$$

are calculable in terms of the derivatives of E,F,G. Now we have

$$\frac{\partial F}{\partial v} = \sum x'_u x''_{v^2} + \sum x'_v x''_{uv} ,$$

$$\frac{\partial G}{\partial u} = 2\sum x'_v x''_{uv} ,$$

hence

$$\vec{M}'_u \vec{M}''_{vv} = \sum x'_u x''_{v^2} = \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} ,$$

and likewise

$$M'_v M''_{uu} = \sum x'_v x''_{u^2} = \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} .$$

There remains the expression τ. We have

$$\frac{1}{2}\frac{\partial^2 G}{\partial u^2} = \sum x''^2_{uv} + \sum x'_v x'''_{u^2 v} ,$$

$$\frac{1}{2}\frac{\partial^2 E}{\partial v^2} = \sum x''^2_{uv} + \sum x'_u x'''_{uv^2} ,$$

$$\frac{\partial^2 F}{\partial u \partial v} = \sum x''_{u^2} x''_{v^2} + \sum x'_u x'''_{uv^2} + \sum x'_v x'''_{u^2 v} + \sum x''^2_{uv} ;$$

hence

$$\tau = \frac{1}{2}\frac{\partial^2 G}{\partial u^2} + \frac{1}{2}\frac{\partial^2 E}{\partial v^2} - \frac{\partial^2 F}{\partial u \partial v} .$$

<u>Remark</u>. We have assumed the existence of the third derivatives of x,y,z.

227. <u>Conjugate directions. Conjugate lines. Lines of curvature. Asymptotic lines.</u>

We say that the two directions $M\tau$, $M\tau'$ of the tangent plane at a point M of a surface, are conjugate when they are conjugate with respect to the

Dupin indicatrix at the point M. They are then conjugate with respect to the asymptotes of this indicatrix, i.e. with respect to the real or imaginary asymptotic tangents. We also say that each of the directions $M\tau$, $M\tau'$ is the conjugate of the other direction. The asymptotic tangents are given in the general case by the equation

$$Ddu^2 + 2D'dudv + D''dv^2 = 0 \quad , \tag{31}$$

which states that the normal curvature is zero. We express the fact that the directions defined by du, dv on one hand, and by δu, δv on the other, are conjugate, by writing

$$Ddu\delta u + D'(du\delta v + dv\delta u) + D''dv\delta v = 0 \quad , \tag{32}$$

i.e., by annihilating the polar form of the second fundamental quadratic form. As the equation (31) is also written as (no. 223)

$$(\vec{M}'_u \wedge \vec{M}'_v)d^2\vec{M} = (\vec{M}'_u, \vec{M}'_v, d^2\vec{M}) = 0 \quad ,$$

the condition for the directions defined by $d\vec{M}$ and $\delta\vec{M}$ to be conjugate is also

$$(\vec{M}'_u \wedge \vec{M}'_v)d\delta\vec{M} = 0 \tag{33}$$

with

$$d\delta\vec{M} = \vec{M}''_{uu}du\delta u + \vec{M}''_{uv}(du\delta v + dv\delta u) + \vec{M}''_{vv}dv\delta v \quad .$$

In equation (33), we can replace the vector product, which is a normal vector to S at M, by any normal vector, $\vec{N}$ say, and write

$$\vec{N}d\delta\vec{M} = 0 \quad . \tag{34}$$

Now we have $\vec{N}d\vec{M} = 0$, hence on differentiating in the direction $\delta\vec{M}$, we have

$$\delta\vec{N}d\vec{M} + \vec{N}d\delta\vec{M} = 0 \quad . \tag{35}$$

The condition of conjugation can therefore be equally stated in the forms (32), (33), (34) or (35)

$$\delta\vec{N}d\vec{M} = 0 \qquad (\text{or} \quad d\vec{N}\delta\vec{M} = 0) \quad .$$

In particular, an *asymptotic tangent*, which is also called an *asymptotic direction*, is its proper conjugate. A principal tangent, which is also called a *principal direction*, is perpendicular to the conjugate direction. When the point is parabolic, the double asymptotic direction forms a pair of conjugate directions with every other direction.

A SYSTEM OF CONJUGATE LINES

Given a line Γ traced on the surface S, we see that at each of its points M, we can define a direction Δ which is conjugate to the direction of

the tangent MT to Γ at the point M. Δ is well defined, except at flat points and parabolic points, at which MT is the asymptotic direction. Δ is determined via the conjugation relationship.

If we consider a *sheaf of curves* Γ such that, through each point M of S, or on a part of S, there passes one and only one curve Γ, then in general, there exists a *conjugate sheaf* formed by curves Γ' such that, through each point M of S, there passes one and only one curve Γ' whose tangent at M is the conjugate direction of the tangent at M to the curve Γ which passes through this point. The determination of this conjugate sheaf is obtained by integrating a differential equation. The relation (32) yields

$$\frac{\delta v}{\delta u} = -\frac{D'dv + Ddu}{D''dv + D'du} ,$$

and, on taking this value into the differential equation

$$h\left(u, v, \frac{\delta v}{\delta u}\right) = 0$$

of the curves Γ, we obtain the differential equation of the curves Γ'.

The curves Γ, Γ' constitute *a system of conjugate lines.*

PARTICULAR CASES OF DEVELOPABLE SURFACES

We know (I,127) that, when $rt - s^2 = 0$, the tangent plane to the surface only depends on one parameter. The surface is the envelope of a plane depending on one parameter; it is a developable surface. Conversely, for every developable surface, we have $rt - s^2 = 0$ (no. 62). Thus: the condition $\mathfrak{M}^2 - \mathfrak{L}\mathfrak{N} = 0$, *or* $D'^2 - DD'' = 0$, *or which is equivalent, the fact that all the points are parabolic, characterizes the developable surfaces.*

In this case, the consideration of the conjugate systems is not so interesting; the rectilinear generators form a conjugate system with any system of curves depending on one parameter. In effect, at each point M, the generator is the intersection of the surface and the tangent plane; it is a double asymptotic direction and it is a conjugate direction of every tangent at M.

THE CONDITION FOR THE COORDINATE CURVES TO FORM A CONJUGATE SYSTEM

In order for the lines $u = \text{const.}$, and $v = \text{const.}$, to form a conjugate system, *it is necessary and sufficient for the condition (32) to be satisfied for* $du = 0$ *and* $\delta v = 0$, *hence for* $D' \equiv 0$.

When x and y are parameters, the conjugation relationship becomes

$$r dx\delta x + s(dx\delta y + dy\delta x) + t dy\delta y = 0 , \tag{36}$$

which can be written as

$$dp\delta x + dq\delta y = 0$$

[which also arises out of the equation (35)]. The condition for the sections through the planes $x = \text{const.}$ and $y = \text{const.}$ to form a conjugate system, is therefore $s \equiv 0$. It follows that z is of the form $z = g(x) + h(y)$. The sections through the planes $x = \text{const.}$ are all equal curves and the surface can be engendered by the transformation of these curves; it is *a surface of translation*. It is one of two different types.

ASYMPTOTIC LINES

These are lines of the surface which, at each of their points M, are tangent to one of the asymptotic tangents at this point. They have as their differential equation

$$Ddu^2 + 2D'dudv + D''dv^2 = 0 \quad ,$$

an equation solvable for $\frac{dv}{du}$ or $\frac{du}{dv}$. We obtain, in general, two sheaves of curves.

Each of these sheaves is its proper conjugate. Following (35), the equation of the asymptotes is

$$d\vec{N}d\vec{M} = 0 \quad ;$$

when x and y are parameters, we have

$$dpdx + dqdy = 0 \quad .$$

In the case of a developable surface, the two systems of asymptotic lines are identified; the asymptotic lines are the generators.

LINES OF CURVATURE

These are lines, which at each of their points, are tangent to one of the principal directions at this point. Following the calculation of no. 225, they are given by the equation

$$\frac{\mathfrak{M}du + \mathfrak{N}dv}{Fdu + Gdv} = \frac{\mathfrak{L}du + \mathfrak{M}dv}{Edu + Fdv} \quad ,$$

or, on replacing $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ by the proportional numbers D, D', D'', by

$$(D'du + D''dv)(Edu + Fdv) - (Ddu + D'dv)(Fdu + Gdv) = 0 \quad . \tag{37}$$

They form a conjugate system, since the principal directions are conjugate. (It is always assumed that, in the case of a developable surface, we assign a different role to the two systems: one is formed by rectilinear generators and the other by their orthogonal trajectories; the generators alone are conjugate to their orthogonal trajectories.) They are rectangular. *These two properties characterize them.*

Remark. It is presupposed that we do not have

$$\frac{D}{E} = \frac{D'}{F} = \frac{D''}{G}$$

throughout. This is a case in which, given that every point is umbilic, every line of S is a line of curvature.

In order for the coordinate lines to be lines of curvature, it is necessary and sufficient that they should be orthogonal, $F = 0$, *and the conjugates* $D' = 0$.

228. The geometric definition of the conjugate lines. The theorem of Koenigs. The geometric definition of the asymptotic lines and the lines of curvature.

Let us consider a line Γ of the surface S. The tangent planes to S at the points M of Γ depend on a parameter; they have a developable surface as an envelope. The generator of this surface, that passes through M, has an envelope that is the edge of regression C of the developable surface; it is tangent to the curve C at a point P. Conversely, if to each point M of Γ there corresponds a tangent MP which has an envelope C when M varies, then the developable surface engendered by these tangents MP, contain Γ and is circumscriptive to S along this curve. Let us call $d\vec{M}$ the differential of $\vec{M}$ when the point M describes Γ and let us take at M, a second tangent

$$\delta\vec{M} = \vec{M}'_u\delta u + \vec{M}'_v\delta v \quad .$$

We intend to determine the ratio $\frac{\delta u}{\delta v}$ as a function of the parameter on which M depends, in such a way that this second tangent has an envelope. It must be possible to determine k, in order that the curve defined by

$$\vec{P} = \vec{M} + k\delta\vec{M}$$

admits MP as a tangent at P. We must have

$$d\vec{P} = d\vec{M} + kd\delta\vec{M} + \delta\vec{M}dk = \lambda\delta\vec{M} \quad ,$$

hence on scalar multiplying a normal vector to S at the point M by $\vec{N}$, which is such that $\vec{N}\delta\vec{M} = 0$, $\vec{N}d\vec{M} = 0$, we see that it is necessary and sufficient that

$$\vec{N}d\delta\vec{M} = 0$$

(k will then be determined by the condition $kd\delta\vec{M} = -d\vec{M} + \lambda'\delta\vec{M}$). The resulting condition is the condition (34) which asserts that the directions $d\vec{M}$ and $\delta\vec{M}$ are conjugate. Consequently:

The Theorem of Dupin. *In order for the directions* $d\vec{M}$ *and* $\delta\vec{M}$ *to be conjugate, it is necessary and sufficient that* $\delta\vec{M}$ *is to be taken by the generator of a developable surface circumscriptive to the surface* S *along a curve* Γ *tangent at* M *to the direction* $d\vec{M}$. *The curve* Γ *is arbitrary.*

From the point of view of applications, a very useful result may be deduced from this theorem:

The Theorem of Koenigs. *Consider a surface* S *and a line* Δ. *Consider also the contact curves* Γ *of the cones circumscriptive to* S *whose vertices are the points of the line* Δ *and the curve sections* Γ' *of the surface* S *through the planes passing through* Δ. *The curves* Γ *and* Γ' *form a conjugate system.*

In effect, if M is a point of a curve Γ', then the tangent to Γ' at M intersects Δ at a point C; the cone with vertex C tangent to S is circumscriptive to S along a curve Γ which passes through M. The line CM is a generator of a developable surface circumscriptive to S; it is conjugate to the tangent to Γ at the point M.

The line Δ could be at infinity.

THE GEOMETRIC DEFINITION OF THE ASYMPTOTIC LINES

Let Γ be an arc of an asymptotic line along which the curvature is nonzero. At a point M of Γ, the curvature of the section of the surface S through the normal plane to S, tangent to Γ, is zero. The osculating plane to Γ at M is therefore tangent to S, otherwise, the curvature of Γ at the point M would be zero on account of the first theorem of no. 223.

Conversely, let us assume that, along an ordinary curve Γ of S, the osculating plane to Γ is constantly tangent to S. Γ will be an asymptotic line. For it is impossible that throughout an arc of Γ, the tangent to Γ is not one of the asymptotic tangents. In effect, if such an arc were to exist, then Meusnier's theorem would show that R, the radius of curvature of Γ, would be zero throughout this arc, which cannot be the case. Γ will therefore be tangent at each point to an asymptotic tangent, with respect to the continuity. Thus:

The asymptotic lines are the lines along which the osculating plane is constantly tangent to the surface.

THE GEOMETRIC DEFINITION OF THE LINES OF CURVATURE

Let Γ be a line of curvature. At each point M of Γ, the conjugate direction of the tangent to Γ is a normal to Γ. On account of the Dupin theorem, these normals to Γ engender a developable surface; similarly for the case of the normals to S along Γ, following the property of the developments of a skew curve (no. 213). Conversely, if the normals to the surface S at the points M of a curve Γ have an envelope, then the tangents to S which are normal to Γ at its various points M also have an envelope and they engender a developable. At each point M, the tangent to Γ has for a conjugate, the direction which is perpendicular to it. The tangent to Γ is therefore a principal direction; Γ is a line of curvature. Consequently:

The lines of curvature of a surface S *are the lines along which the normal to the surface has an envelope.*

229. Properties of the lines of curvature. Normalities. The theorem of Joachimstal. Umbilics. Triply orthogonal systems.

Let Γ be a line of curvature, M one of its points, MT the tangent to Γ at M and MN the normal to S at the point M. The section of the surface through the plane MTN, normal to S at the point M, has at this point, a center of curvature I which is found on the axis of curvature of Γ, by virtue of Meusnier's theorem. Given that the point of contact of the normal MN with its envelope γ is also on the axis of curvature, following the properties of the developments (no. 213), it coincides with I. Hence:

If Γ *is a line of curvature, the points of contact of the normals to* S *at the points* M *of* Γ *with their envelope* γ, *are the centers of principal curvature of* S, *relative to the points* M *and to the normal sections of* S *tangent to* Γ.

Let γ be the envelope of the normals to S along Γ. When Γ is displaced continuously, the lines γ engender a surface Σ being the locus of one of the centers of principal curvature to the various points of S; the normals are tangent to Σ. If Γ' is the second line of curvature passing through M (there is only one of them, apart from Γ, if M is not umbilic), then the normals to S along Γ' have an envelope γ' (fig. 75), where the point of contact of MN with γ' is distinct from I. When Γ' is varied continuously, γ', engenders a surface Σ', to which the normals at S are

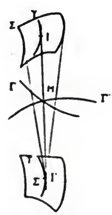

Fig. 75.

also tangents. *The normals to* S *which depend on two parameters form a congruence of lines.* The surfaces Σ and Σ' are called *the sheets of the focal surface* of this congruence, they constitute *the surface of the centers.* The developable surfaces engendered by the normals to S along the lines Γ or Γ' are called the *normalities*; they are tangent to the surfaces Σ and Σ', respectively. The planes tangent at the points I' and I to Σ' and Σ, pass through the tangents MT and MT' respectively to Γ and Γ' at the point M; *they are therefore rectangular.* In effect, the tangent plane to Σ at the point I, is determined by MI and by the tangent to the curve described by I when M is displaced on the line Γ', it is therefore the

tangent plane to I to the normality defined by Γ'. The tangent plane to Σ' at I' will likewise, be the tangent plane to the normality defined by Γ. Now these tangent planes are rectangular. Similarly, this is the case if, when γ or γ' reduce to a point, Σ or Σ' is a curve.

Later (no. 253), we shall see that these properties characterize the congruences of normals to a surface. *They are true only when* M *is displaced on a part of* S *not containing an umbilic.* At an umbilic, the differential equation of the lines of curvature presents a singularity which we have not studied as yet, but in simple cases, the disposition of the lines of curvature ending at such a point, will result from a direct study or by continuity considerations.

AN EXAMPLE OF SURFACES OF REVOLUTION

Let us consider the case of a surface of revolution S. At a point M which is not on the axis Δ, where the plane $M\Delta$ is a plane of symmetry, the indicatrix is symmetric with respect to this plane and the tangent to the parallel and the tangent to the meridian, are certainly principal tangents. The meridians Γ and the parallels Γ' are lines of curvature. If P is the projection of M onto the axis Δ and I' the point where the normal to Γ at M intersects Δ, then I' is the center of curvature at M of the normal section containing the tangent to Γ'. This is a consequence of either Meusnier's theorem, or from the fact that the normals to S along Γ, intersect the axis at the same point, as we can see (fig. 76).

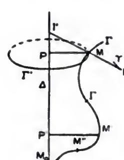

Fig. 76.

Here γ' reduces to the point I'. The corresponding sheet Σ' of the focal surface, is a part of the axis Δ. As for the second center of principal curvature I at M, this is the center of curvature at M of the meridian Γ; γ is the developed plane of Γ and Σ is the surface of revolution engendered by γ, turning about Δ. The plane $\Delta I'M$ is perpendicular to the tangent plane to Σ at the point I.

If at a point M' of Γ, not situated on Δ, the tangent is parallel to Δ and if, moreover, the center of curvature of Γ at M' is seen to be identified with P', the projection of M' on Δ, then M' is an umbilic. In all, we will obtain a parallel whose points will be umbilics and through

each point of this parallel, there will pass only one other line of curvature if M' is an isolated point having the indicated property.

The points of inflexion of Γ provide parallels whose points are parabolic points; it is the same for the points of Γ at which the tangent is perpendicular to Δ. A point of inflexion M'' of Γ at which the tangent is perpendicular to Δ provides a parallel for which every point is a flat point; through each point of this parallel, there passes two lines of curvature, Γ and Γ'.

If Γ intersects the axis Δ at an ordinary point M_0 and if the tangent is perpendicular to Δ at this point M_0, then M_0 is an umbilic, the lines of curvature Γ end at M_0 and the lines Γ' encircle this point.

The Theorem of Joachimstal. *Two surfaces* S *and* S' *having in common a line of curvature* Γ, *intersect at a constant angle along* Γ. *Conversely, if* S *and* S' *intersect at a constant angle along a curve* Γ *which is a line of curvature for* S, *then* Γ *is also a line of curvature for* S'.

This is a consequence of the theorem relating to the developments of a curve Γ: any two developable surfaces engendered by the normals to Γ intersect at a constant angle along Γ (no. 213). If S and S' have a line of curvature Γ in common, then the normals MN and MN' to S and S' at any point M of Γ, make a constant angle. Conversely, if Γ is a line common to S, and S' is a line of curvature of S, then MN engenders a developable surface. If the NMN' is constant, MN' also engenders a developable surface, and Γ is a line of curvature of S'.

EXAMPLES

I. *If a surface* S *is intersected at a constant angle by a plane* π *along a line* Γ, *then* Γ *is a line of curvature of* S. For this can be regarded as a line of curvature of π (the normals to π along Γ form a cylinder).

II. *If a surface* S *is intersected at a constant angle by a sphere along a curve* Γ, *then* Γ *is a line of curvature of* S. For it is the line of curvature of the sphere; every line of a sphere is in effect a line of curvature since the normals to the sphere along this line form a cone.

AN APPLICATION TO THE STUDY OF UMBILICS

Let us assume that all the points of a part S of a surface (constituting a two-dimensional domain) are umbilical points. If M_0 is a point of S and M_0N_0 the normal at this point, then apply plane passing through M_0N_0 intersects S along a line of curvature, and hence at a constant angle, following Joachimstal's theorem. This is a right-angle since it is so at M_0. Therefore, the normals to S at the points of the plane section in question are in the plane of this section. The normals to S intersect

M_0N_0. They will also intersect the normal to S at another point M_1, i.e. they will pass through the point of intersection of M_0N_0 and M_1N_1. Let C be this point at a finite or infinite distance. The sections of S through the planes passing through C, are the arcs of a circle of center C since their normals tend to the point C, unless C happens to be at infinity; this is the case where we would obtain a straight line. S is part of a sphere or part of a plane. Thus: *When all the points of a part of* S *constituting a domain, are umbilics without being flat points, then this part is spherical. If one of these points is a flat point, then this part of* S *is planar.*

Remark. When S is analytic, the fact that a piece of S is a plane or a sphere implies that S is a plane or a sphere. But if S is not analytic, it is clear that one piece of S could be part of a plane, another that of a sphere, without S being entirely deprived of the properties of derivation which are assumed.

TRIPLY ORTHOGONAL SYSTEMS. THE THEOREM OF DUPIN

Let us assume that in a domain Δ of the space, we have defined a system of three sheaves of surfaces $S_1(\lambda)$, $S_2(\mu)$, $S_3(\nu)$ depending on the parameters λ,μ,ν respectively, such that through each point M of Δ, there passes one and only one from each sheaf and that these surfaces intersect orthogonally in pairs. *We say that such a system is a triply orthogonal system.*

The orthogonality conditions imply that the surfaces intersect in pairs along curves which themselves form a triply orthogonal system. By restricting Δ, if needs be, we can assume that, to a system of values λ,μ,ν, there corresponds a single point of intersection of S_1,S_2,S_3, M say. The vector $\vec{M}(\lambda,\mu,\nu)$ allows us to define the surfaces S_1,S_2,S_3 by assuming λ, or μ, or ν to be constant, and their curves of intersection: C_3 of S_1 and S_2 (λ and μ constant), C_1 of S_2 and S_3 ($\mu = \text{const.}$, $\nu = \text{const.}$), C_2 of S_3 and S_1 ($\nu = \text{const.}$, $\lambda = \text{const.}$).

At the point $\vec{M}$, the normal to S_1 is defined by $\vec{M}'_\lambda$, the tangents to the curves C_3 and C_2 are given by $\vec{M}'_\nu$ and $\vec{M}'_\mu$; *these two tangents are orthogonal. We are going to show that they are conjugate.* In order to verify this, we need to prove (no. 227) that

$$\vec{M}'_\lambda \vec{M}''_{\mu\nu} = 0 \quad . \tag{38}$$

Now we have

$$\vec{M}'_\lambda \vec{M}'_\mu \equiv 0, \qquad \vec{M}'_\lambda \vec{M}'_\nu \equiv 0, \qquad \vec{M}'_\mu \vec{M}'_\nu \equiv 0 \ ,$$

by differentiating the first identity with respect to ν, the second with respect to μ and the third with respect to λ. We obtain

$$\vec{M}'_\lambda \vec{M}''_{\mu\nu} \equiv -\vec{M}''_{\lambda\nu}\vec{M}'_\mu , \qquad \vec{M}'_\lambda \vec{M}''_{\mu\nu} \equiv -\vec{M}''_{\lambda\mu}\vec{M}'_\nu ,$$

$$\vec{M}''_{\lambda\mu}\vec{M}'_\nu + \vec{M}''_{\lambda\nu}\vec{M}'_\mu \equiv 0 \quad .$$

On combining these three identities, we obtain the equation in (38). Thus, the lines C_3 and C_2 situated on S_1 are orthogonal and are conjugate; these are the lines of curvature. From this we deduce Dupin's theorem: *The surfaces of a triply orthogonal system intersect along their lines of curvature.*

Example. An ellipsoid belongs to a triply orthogonal system formed by the ellipsoids and hyperboloids with one and two sheets which are homofocal to it; its lines of curvature are therefore curves of intersection with the homofocal hyperboloids. These are the biquadrics defined by the equations (9) of no. 221.

We shall discuss some other applications of Dupin's theorem, at a later stage. Let us remark here, that the lines of curvature of the paraboloids which are not those of revolution, are also obtained from a triply orthogonal system defined by

$$\frac{x^2}{p-\lambda} + \frac{y^2}{q-\lambda} - 2z + \lambda = 0 \qquad (p \neq q) \quad .$$

230. Spherical representation.

If S is part of an ordinary surface, $H \neq 0$, then at each point $M(u,v)$, we can define the unit vector of the normal by

$$\vec{N} = \frac{\vec{M}'_u \wedge \vec{M}'_v}{H}$$

(no. 223). If, through a fixed point O (which can be assumed to be the origin of the axes), we take a vector equipollent to $\vec{N}$, then the extremity μ of this vector remains on the sphere σ of radius 1 and center O. If the surface S is developable, then the point μ describes a curve which is the indicatrix of the binormals of the edge of regression. We shall put this case aside.

When M describes a curve Γ on S, μ, in general, describes a curve γ of the sphere σ. *This curve* γ *is the spherical image of* Γ. If M describes a small closed curve Γ on S, then μ, in general, describes a small closed curve γ on σ and, if Γ remains sufficiently near to a point M_0, there exists a bijective correspondence between the points of S inside Γ and the points of σ inside γ. *The interior of* Γ *is represented bijectively on the interior of* γ. It is clear that, to the corresponding points M, μ, the tangent planes to S and σ are parallel. In the particular case where S is a convex surface, the correspondence between the

points M and the points μ of the spherical representation, is bijective on all the surface S.

<u>Remark</u>. If S is tangent to a plane along a line (the case of extreme parallels in a torus), then to this line, there corresponds only one point in the spherical representation. We have

$$d\vec{\mu} = d\vec{N} = \vec{N}'_u du + \vec{N}'_v dv \quad .$$

The vector $\vec{N}$ being unitary, $\vec{N}'_u$ and $\vec{N}'_v$ are perpendicular to it; they are parallel to the tangent plane S at the point M, and hence are of the form

$$\vec{N}'_u = m\vec{N}'_u + n\vec{M}'_v \ , \qquad \vec{N}'_v = m'\vec{N}'_u + n'\vec{M}'_v \tag{39}$$

where m, n, m', n' are calculated by scalar multiplication by $\vec{M}'_u$ and $\vec{M}'_v$.

$$mE + nF = \vec{N}'_u\vec{M}'_u \ , \qquad mF + nG = \vec{N}'_u\vec{M}'_v \ ;$$

$$m'E + n'F = \vec{N}'_v\vec{M}'_u \ , \qquad m'F + n'G = \vec{N}'_v\vec{M}'_v \ .$$

Now we have

$$\vec{N}\vec{M}'_u = 0, \qquad \vec{N}'_u\vec{M}'_u = -NM''_{uu} = -\frac{(\vec{M}'_u, \vec{M}'_v, \vec{M}''_{uu})}{H}$$

$$= -\mathcal{L} \quad ,$$

and similar equalities. Hence

$$mE = nF = -\mathcal{L} \ , \qquad mF + nG = -\mathfrak{M}$$

$$m'E + n'F = -\mathfrak{M}, \qquad m'F + n'G = -\mathcal{N} \ .$$

These relations yield m, n, m', n'. We have

$$\left.\begin{aligned} mH^2 &= \mathfrak{M}F - \mathcal{L}G \ , \qquad & nH^2 &= \mathcal{L}F - E\mathfrak{M} \\ m'H^2 &= \mathcal{N}F - \mathfrak{M}G \ , \qquad & n'H^2 &= \mathfrak{M}F - E\mathcal{N} \end{aligned}\right\} \tag{40}$$

It follows that

$$(mn' - m'n)H^2 = \mathcal{L}\mathcal{N} - \mathfrak{M}^2 \quad . \tag{41}$$

<u>The Gauss formula</u>. The element of area on the surface S is

$$dS = Hdudv \ , \qquad H = |\vec{M}'_u \wedge \vec{M}'_v| \quad .$$

On the sphere σ, the corresponding element of area is

$$d\sigma = |\vec{N}'_u \wedge \vec{N}'_v| dudv \quad .$$

In the second member, we have, on account of (39),

$$\vec{N}'_u \wedge \vec{N}'_v = (mn' - m'n)\vec{M}'_u \wedge \vec{M}'_v \quad . \tag{42}$$

It follows that

$$\frac{d\sigma}{dS} = |mn' - m'n| = \frac{|\mathcal{L}\mathcal{N} - \mathcal{M}^2|}{EG - F^2} \quad ,$$

and, on applying formula (29) which gives the total curvature, we obtain the Gauss formula

$$\frac{d\sigma}{dS} = \left|\frac{1}{R_1 R_2}\right| \quad . \tag{43}$$

The total curvature at a point M *of a surface is the limit of the ratio of the area of the spherical representation of the interior of a small closed curve* Γ *about* M, *to the area of the interior of* Γ, *when* Γ *tends toward* M.

Following the equality in (42), the normal vectors to S and σ at the points M and μ,

$$\vec{M}'_u \wedge \vec{M}'_v \ , \qquad \vec{N}'_u \wedge \vec{N}_v \tag{44}$$

are of the same or opposite sense depending on whether $mn' - m'n$ is positive or negative respectively, i.e. whether the total curvature is positive or negative respectively [for on account of (41) and (29), we have $R_1 R_2 (mn' - m'n) = 1$].

If the vectors (44) are of the same sense, then about the normals to the points M and μ, the sense of rotation of $\vec{M}'_u$ towards $\vec{M}'_v$ on one hand, and that of $\vec{N}'_u$ towards $\vec{N}'_v$, on the other, are the same. Under these conditions, when a point describes a small closed curve about M in the sense of $\vec{M}'_u$ towards $\vec{M}'_v$ then the corresponding point will turn about μ in the same sense. *The senses of rotation about* M *and* μ, when a point of S describes a closed curve about M and the spherical image undergoes the corresponding displacement, *are therefore the same when the total curvature is positive. They are opposite if the total curvature is negative.*

THE CORRESPONDENCE BETWEEN THE DIRECTIONS ON S AND ON σ

On account of the relationships in (39), the relation between a direction passing through M on S and the corresponding direction passing through μ, is homographic. In order to study this relation, it suffices to observe that to the direction $d\vec{M}$ of S, there corresponds on σ, the direction $d\vec{N}$. Now, given that $\delta\vec{M}$ is the conjugate direction of $d\vec{M}$, we know that $d\vec{N}\delta\vec{M} = 0$ [formula (35)]. Hence: *To a given direction* $d\vec{M}$ *at the point* M, *the spherical representation assigns the perpendicular direction to the conjugate direction* $\delta\vec{M}$.
It follows that:

A principal direction is parallel to its spherical representation.
An asymptotic direction is perpendicular to its spherical representation.

II. GEODESIC CURVATURE AND TORSION
GEODESICS - ISOMETRIC SURFACES

231. The trihedron of Darboux-Ribaucour. Geodesic torsion and curvature.

Let us consider an ordinary point M $(H \neq 0)$ on a surface S defined by $\vec{M}(u,v)$. The unit normal is always defined by

$$\vec{N} = \frac{1}{H}\vec{M}'_u \wedge \vec{M}'_v \quad .$$

Given a curve Γ on S, passing through M and admitting this point as an ordinary point, *the trihedron of Darboux-Ribaucour relative to* S *and* Γ is the trihedron defined by the unit vector $\vec{t}$ of the tangent to Γ at M, a unit vector $\vec{g}$ such that

$$\vec{g} = \vec{N} \wedge \vec{t} \quad ,$$

and the vector $\vec{N}$. The trihedron $\vec{t}, \vec{g}, \vec{N}$ is a direct trihedron. We shall always denote by $\vec{n}$ and $\vec{b}$ respectively the unit vectors of the principal normal and of the binormal of Γ. In figure 77, $\vec{g}, \vec{N}, \vec{n}, \vec{b}$ are in the plane of the figure; $\vec{t}$ normal to this plane, is foremost. One passes from the

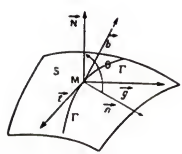

Fig. 77.

Frenet-Serret trihedron to that of Darboux-Ribaucour by a rotation about $\vec{t}$. It is deduced from θ, which is the measure in the plane $\vec{g}, \vec{N}$ (or $\vec{n}, \vec{b}$) of the angle through which $\vec{n}$ must be turned, in order to position it on $\vec{N}$. Following the well-known trigonometric formulae, we therefore have

$$\vec{N} = \vec{n}\cos\theta + \vec{b}\sin\theta$$

and

$$\vec{g} = \vec{N} \wedge \vec{t} = -\vec{b}\cos\theta + \vec{n}\sin\theta \quad .$$

Similarly, since the angle $\vec{n}$ makes with $\vec{g}$, is $\frac{\pi}{2} - \theta$, we have

$$\vec{n} = \vec{g}\sin\theta + \vec{N}\cos\theta \ , \qquad \vec{b} = -\vec{g}\cos\theta + \vec{N}\sin\theta \quad .$$

The formulae of Frenet-Serret, then become

$$\frac{d\vec{t}}{ds} = \frac{\vec{n}}{R} = \frac{\sin\theta}{R}\vec{g} + \frac{\cos\theta}{R}\vec{N},$$

$$\frac{d\vec{g}}{ds} = (\vec{b}\sin\theta + \vec{n}\cos\theta)\frac{d\theta}{ds} - \frac{d\vec{b}}{ds}\cos\theta + \frac{d\vec{n}}{ds}\sin\theta$$

$$= \frac{d\theta}{ds}\vec{N} + \frac{\vec{n}}{T}\cos\theta + \left(\frac{\vec{b}}{T} - \frac{\vec{t}}{R}\right)\sin\theta = \left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{N} - \frac{\vec{t}}{R}\sin\theta\ ;$$

$$\frac{d\vec{N}}{ds} = (-\vec{n}\sin\theta + \vec{b}\cos\theta)\frac{d\theta}{ds} + \frac{d\vec{b}}{ds}\sin\theta + \frac{d\vec{n}}{ds}\cos\theta$$

$$= -\frac{d\theta}{ds}\vec{g} - \frac{\vec{n}}{T}\sin\theta + \left(\frac{\vec{b}}{T} - \frac{\vec{t}}{R}\right)\cos\theta = -\left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{g} - \frac{\vec{t}}{R}\cos\theta\ .$$

We have the transformed formulae

$$\frac{d\vec{t}}{ds} = \frac{\sin\theta}{R}\vec{g} + \frac{\cos\theta}{R}\vec{N},$$
$$\frac{d\vec{g}}{ds} = -\frac{\vec{t}}{R}\sin\theta + \left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{N}, \tag{45}$$
$$\frac{d\vec{N}}{ds} = -\frac{\vec{t}}{R}\cos\theta - \left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{g}\ .$$

As $\frac{d\vec{N}}{ds}$, $\vec{t}$, $\vec{g}$, only depend on the tangent at M, then *the quantities*

$$\frac{\cos\theta}{R}, \qquad \frac{1}{T} + \frac{d\theta}{ds}$$

are the same for all curves of S *having the same tangent at* M. We know this already for the first quantity; this is Meusnier's theorem; $\frac{\cos\theta}{R}$ is the curvature of the normal section, $\frac{1}{R_N}$, where R_N has a sign.

The geodesic curvature of Γ *at the point* M, *is the quantity*

$$\frac{1}{R_g} = \frac{\sin\theta}{R},$$

and the geodesic torsion of Γ *at* M *is the number*

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds}\ .$$

By introducing these numbers, the formulae in (45) are written

$$\frac{d\vec{t}}{ds} = \frac{\vec{g}}{R_g} + \frac{\vec{N}}{R_N}, \qquad \frac{d\vec{g}}{ds} = -\frac{\vec{t}}{R_g} + \frac{\vec{N}}{T_g}$$

$$\frac{d\vec{N}}{ds} = -\frac{\vec{t}}{R_N} - \frac{\vec{g}}{T_g}\ . \tag{46}$$

232. The calculation of the geodesic torsion. Bonnet's formula. Enneper's formula.

The geodesic torsion is obtained by scalar multiplication of the second formula in (46) by $\vec{N}$, or the third formula by $\vec{g}$. We have

$$\frac{1}{T_g} = \vec{N}\cdot\frac{d\vec{g}}{ds} = -\vec{g}\,\frac{d\vec{N}}{ds}$$

and, since $\vec{g} = \vec{N}\wedge\vec{t}$,

$$\frac{1}{T_g} = \frac{(\vec{t},\vec{N},d\vec{N})}{ds} = \frac{(d\vec{M},\vec{N},d\vec{N})}{ds^2} \quad . \tag{47}$$

But, on account of the property of the double vector product, we can also write

$$H\vec{g} = (\vec{M}'_u\wedge\vec{M}'_v)\wedge\vec{t} = (\vec{t}\vec{M}'_u)\vec{M}'_v - (\vec{t}\vec{M}'_v)\vec{M}'_u$$

and, as

$$\vec{t}ds = d\vec{M} = \vec{M}'_u du + \vec{M}'_v dv \quad ,$$

we obtain

$$H\vec{g}ds = (Edu + Fdv)\vec{M}'_v - (Fdu + Gdv)\vec{M}'_u \quad . \tag{48}$$

On the other hand,

$$d\vec{N} = \vec{N}'_u du + \vec{N}'_v dv \quad , \tag{49}$$

and, we have seen (no. 230), that

$$\vec{N}'_u\vec{M}'_u = -\mathcal{L} \; , \qquad \vec{N}'_u\vec{M}'_v = \vec{N}'_v\vec{M}'_u = -\mathfrak{M} \; , \qquad \vec{N}'_v\vec{M}'_v = -\mathcal{N} \quad ;$$

the product of (48) and (49) therefore yields

$$\begin{aligned}\vec{g}d\vec{N}Hds = {} & (Fdu + Gdv)(\mathcal{L}du + \mathfrak{M}dv) \\ & - (Edu + Fdv)(\mathfrak{M}du + \mathcal{N}dv) \quad .\end{aligned}$$

It follows that

$$\frac{1}{T_g} = \frac{1}{Hds^2}\left\{(Edu + Fdv)(\mathfrak{M}du + \mathcal{N}dv) - (Fdu + Gdv)(\mathcal{L}du + \mathfrak{M}dv)\right\} \quad . \tag{50}$$

Remark. This formula shows that the sign of T_g does not depend on the sense of direction chosen for the curve in question.

Bonnet's formula. If we take the point M to be the origin O and if for Ox, Oy, we take the principal tangents and for Oz the positive normal, then by assuming that the parameters are x and y, we will have:

$$E = G = 1 \; , \qquad F = \mathfrak{M} = 0 \; , \qquad H = 1 \; , \qquad \mathcal{L} = r \; , \qquad \mathcal{N} = t \; ,$$

and

$$\frac{1}{T_g} = \frac{(t-r)dxdy}{ds^2} .$$

On setting $dy = tg\omega dx$, we see from no. 224 that the normal curvature is then given by the Euler formula,

$$\frac{1}{R_N} = r \cos^2\omega + t \sin^2\omega .$$

The geodesic torsion is

$$\frac{1}{T_g} = (t-r)\sin\omega\cos\omega = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\omega\cos\omega , \tag{51}$$

where R_1 *and* R_2 *are the radii of principal curvature corresponding to the sections* $\omega = 0$ *and* $\omega = \frac{\pi}{2}$. THIS IS BONNET'S FORMULA.

THE CASE OF ASYMPTOTIC LINES

In order for a line Γ to be asymptotic, the osculating plane is tangent to the surface S. We can assume that the sense of direction is chosen in such a way that the trihedron $\vec{t}, \vec{n}, \vec{N}$ is direct; then the Frenet-Serret trihedron coincides with that of Darboux-Ribaucour and we have $\theta = \frac{\pi}{2}$. *The geodesic torsion is equal to the torsion of* Γ*; the geodesic curvature is equal to the curvature of* Γ. The torsion of an asymptotic line is therefore given by

$$\frac{1}{T} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\omega\cos\omega ,$$

with

$$tg^2\omega = -\frac{r}{t} = -\frac{R_2}{R_1} , \qquad tg\omega = \pm\sqrt{-\frac{R_2}{R_1}} .$$

We therefore have

$$\frac{1}{T} = \pm\sqrt{-\frac{R_2}{R_1}} \cdot \frac{1}{R_2} .$$

The torsions of two asymptotic lines passing through M (the case of one hyperbolic point) *are opposite. We have*

$$T = \pm\sqrt{-R_1R_2} .$$

THIS IS ENNEPER'S FORMULA.

233. Geodesic curvature and the center of geodesic curvature. Bonnet's theorem.

The calculation of the geodesic curvature evolves from the scalar multiplication of the first formula (46) by $\vec{g}$. We have

$$\frac{1}{R_g} = \frac{\sin\theta}{R} = \vec{g}\,\frac{d\vec{t}}{ds} \quad ,$$

where g is given by the formula (48). On the other hand, by assuming that we have taken the arc of the curve Γ as parameter, we have

$$\vec{t} = \vec{M}'_u u + \vec{M}'_v v' \quad ,$$

$$\frac{d\vec{t}}{ds} = \vec{M}''_{uu} u'^2 + 2\vec{M}''_{uv} u'v' + \vec{M}''_{vv} v'^2 + \vec{M}'_u u'' + \vec{M}'_v v'' \quad .$$

On taking the product of this last equality and the two members of (48) and taking account of the values of $\vec{M}'_u\vec{M}''_{vv}$ and $\vec{M}'_v\vec{M}''_{uu}$ calculated in the course of the proof of the Gauss theorem of no. 226, we immediately obtain

$$\frac{H}{R_g} = H^2(u'v'' - u''v') + \tag{52}$$

$$\begin{vmatrix} Eu' + Fv' & \frac{1}{2} E'_u u'^2 + E'_v u'v' + \left(F'_v - \frac{1}{2} G'_u\right) v'^2 \\ Fu' + Gv' & \left(F'_u - \frac{1}{2} E'_v\right) u'^2 + G'_u u'v' + \frac{1}{2} G'_v v'^2 \end{vmatrix}$$

We see that the geodesic curvature, as in the case of the total curvature, only depends on E, F, G and their derivatives, and also on the elements of Γ. This is a veritable geodesic element (see no. 236).

THE CENTER OF GEODESIC CURVATURE

The center of geodesic curvature of the curve Γ at the point M, is the point of intersection of the axis of curvature of Γ at the point M, with the tangent plane to S at M. It is therefore situated on the axis taken by $\vec{g}$, its distance to M is R_g.

Bonnet's Theorem. *Let us consider a simply connected region of an ordinary surface* S, *bounded by a simple closed curve* Γ. *If* R_1 *and* R_2 *are the radii of principal curvature at a point* M *of* S, dS *the element of area at this point and* R_g *the radius of geodesic curvature at a point of* Γ, *then we have*

$$\int_\Gamma \frac{ds}{R_g} = 2\pi - \iint_\Delta \frac{dS}{R_1 R_2} \quad ,$$

where the integral is taken on Γ *in the direct sense with respect to the positive normal defined on* S.

We shall first of all prove the theorem by taking a sufficiently small domain on Δ in order that it corresponds bijectively to its spherical representation. When M describes Γ, then the vector $\vec{g}$ is directed towards the interior of Γ. The first member of the equality in (53) is

$$\int_\Gamma \vec{g}\, d\vec{t}\ .$$

When M describes Γ, then the point μ of the spherical representation describes a closed curve γ. We assume that the interior of γ corresponds bijectively to Δ. This implies that the total curvature is of constant sign in Δ. In effect, if, on both sides of a line Γ' of S, the total curvature is of opposite sign, then the correspondence between S and the sphere σ is of a different sense; direct on one side and inverse on the other (no. 230): the spherical representation is recovered where γ has a double point (the case of the torus) (fig. 78).

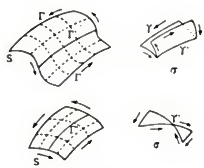

Fig. 78

If the total curvature is positive, γ *follows* Γ *in the direct sense with respect to* $\vec{N}$; *if the total curvature is negative, then* γ *passes in the inverse sense.*

Let us take two unit, rectangular vectors $\vec{\tau}\,\vec{\rho}$ in the tangent plane to σ at the point μ; these are derivable functions of the position of μ, i.e. of s, such that $\vec{\tau}$, $\vec{\rho}$, $\vec{N}$ form a direct trihedron. We have

$$\vec{t} = \vec{\tau}\cos\omega + \vec{\rho}\sin\omega\ , \qquad \vec{g} = -\vec{\tau}\sin\omega + \vec{\rho}\cos\omega\ ,$$

where the angle ω is differentiable with respect to s. It follows that

$$\vec{g}\, d\vec{t} = d\omega + \vec{\rho}\, d\vec{\tau} \qquad ,$$

and we have

$$\int_\Gamma \vec{g}\, d\vec{t} = \int_\gamma d\omega + \int_\gamma \vec{\rho}\, d\vec{\tau} \qquad , \tag{54}$$

where the sense of direction on γ is direct if $R_1R_2 > 0$, inverse if $R_1R_2 < 0$, with respect to the normal $\vec{O\mu}$.

When μ describes γ, the angle ω varies and re-attains its initial value at a multiple of 2π since the vectors $\vec{\tau}$ and $\vec{t}$ recover their initial position. The first integral in (54) is equal to $2\pi m$, where m is an integer. Let us consider the second by assuming, for example, $R_1R_2 > 0$. Let μ_0 be a point outside of the domain δ of the sphere σ which corresponds to Δ. We can assume μ_0 to be on $0z$ and we can take, on the sphere, spherical coordinates: ϕ the angle of the plane $\mu 0z$ with $0x$ and ψ the angle of 0μ with $0z$ (fig. 79). We may allow ourselves to take

$$\vec{\tau} = -\vec{i}\sin\phi + \vec{j}\cos\phi \ ,$$

where $\vec{i}, \vec{j}, \vec{h}$ are unit vectors along the axes $0x, 0y, 0z$.

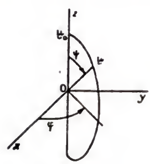

Fig. 79.

Since

$$\vec{N} = 0\vec{\mu} = \vec{i}\cos\phi\sin\psi + \vec{j}\sin\phi\sin\psi + \vec{k}\cos\psi \ ,$$

we have

$$\vec{\rho} = \vec{N}\wedge\vec{\tau} = -\vec{i}\cos\phi\cos\psi - \vec{j}\sin\phi\cos\psi + \vec{h}\sin\psi \ ,$$

hence, since we also have

$$d\vec{\tau} = -(\vec{i}\cos\phi + \vec{j}\sin\phi)d\phi \ ,$$

$$\vec{\rho}d\vec{\tau} = \cos\psi \, d\phi \ .$$

Now

$$\int_\gamma \cos\psi d\phi$$

can be transformed by the Riemann formula into the plane (ψ,ϕ). In this plane, the origin $\phi = 0$, $\psi = 0$ is the image of μ_0; it is outside the curve γ_1 corresponding to γ and γ_1 is taken in the direct sense. For γ can be assumed to be near to μ_0 and ψ takes the role of a radius vector and ϕ that of a polar angle. We therefore have

$$\int_\gamma \cos\psi d\phi = -\iint_\delta \sin\psi \, d\phi d\psi = -\iint_\delta d\sigma \ , \tag{55}$$

where $d\sigma$ is the element of area of the sphere. Now we know that, on account

of the Gauss formula, we have, since $R_1R_2 > 0$,

$$d\sigma = \frac{dS}{R_1R_2} \quad . \tag{56}$$

If $R_1R_2 < 0$, γ is taken in the inverse sense, it is necessary to change the sign of the second and third member of (55), and that of (56). By means of the hypotheses established, we then have in both cases

$$\int \frac{ds}{R_g} = 2\pi m - \iint_\Delta \frac{dS}{R_1R_2} \quad . \tag{57}$$

When we vary Γ continuously, the integrals of both members of this formula, vary continuously, hence m remains fixed. Now, if Γ is a section of S through a cylinder of revolution whose axis is the normal to S at M_0 and the radius ε, we have on taking as the axes M_0x, M_0y and M_0z, identified with the normal, $x = \varepsilon \cos u$, $y = \varepsilon \sin u$,

$$z = \frac{\varepsilon^2}{2}(r \cos^2 u + 2s \sin u \cos u + t \sin^2 u) + \cdots$$

and we can say that the osculating plane at every point of this section tends towards the tangent plane at M_0 when ε tends towards zero. The ratio of the radius of geodesic curvature to ε tends towards 1, the length of the curve is $2\pi\varepsilon$ to within a factor which tends towards 1, and the first member of (57) is near to 2π when the integral of the second member tends towards zero. Hence $m = 1$.

Bonnet's theorem is thus established under the following conditions: *Δ corresponds bijectively to its spherical representation and the total curvature has a constant sign in Δ.*

In order to complete the proof, it would perhaps help to see the statement is modified if Γ has angular points between which it is an ordinary curve. To an angular point of Γ, there corresponds an angular point of γ and at such a point μ, the angle ω which appears in (54), has a discontinuity, and the above argument no longer applies. But let us consider a curve Γ presenting an angular point at M_0 and elsewhere satisfying the other conditions mentioned above. Let α be the angle of the tangents at M_0 taken in the sense of motion on Γ (fig. 80).

Fig. 80.

We can trace a cylinder of revolution of radius ε whose axis is parallel to the normal to S at M_0, tangent to Γ at two points M_1 and M_2 situated on both sides of M_0, and replace Γ, between M_1 and M_2, by the section

M_1M_2 of the surface S by this cylinder. Formula (53) applies to the curve Γ thus modified, Γ' say, and to the domain Δ' which it bounds, and, when ε tends towards zero, the integral of the second member taken in Δ', tends towards the integral taken in Δ.

In the first member, we must replace the integral of $\frac{ds}{R_g}$ taken on the arc M_1M_2 of Γ' by the integral taken on $M_1M_0M_2$; we have

$$\int_{\Gamma'} \frac{ds}{R_g} = \int_{\Gamma} \frac{ds}{R_g} + \int_{M_1\Gamma'M_2} \frac{ds}{R_g} - \int_{M_1M_0M_2} \frac{ds}{R_g} .$$

The last integral tends towards zero with ε. The preceding one, in which the ratio of R_g to ε tends towards 1 following the remark made above, whilst the arc of integration is equivalent to $\alpha\varepsilon$, tends towards α. The formula (53) holds true which the condition of including α in the first member. The calculation also goes through if α is negative (the angle of reentrance). If there are several angular points, we must add the corresponding angles to the first member.

Let us assume then, that given the total curvature has a constant sign in Δ, Δ does not correspond bijectively to its spherical representation (the conditions of the statement of the theorem having been satisfied). We will assume that it is possible to partition Δ into a finite number of domains corresponding bijectively to their spherical representation, by constructing lines which will introduce the angular points. We apply the formula to each partial domain and add up. For example, if there are four domains (fig. 81), we will have in the first member of the formula, the integrals relative to the geodesic curvature which will yield the integral taken on Γ; the others are removed pairwise. The angles 1, 2, 3, 4 will all give a sum 4π and the angle 5 will give 2π. We can indeed see that the formula (53) holds true.

Fig. 81.

We proceed in a similar way if the normal curvature is zero on changing sign on certain lines.

The theorem is thus established and we see that we can add the following corollary:

<u>Corollary</u>. *The Bonnet formula (53) remains true if* Γ *presents* n *angular points, with the condition of adding* $n\pi - \Sigma_1^n \alpha_j$ *to the first member, where the* α_j *are the angles of* Γ *at the angular points measured in the interior of* Δ.

234. Applications of geodesic torsion

Following the Bonnet formula (51) of no. 232, a line whose geodesic torsion is zero at every point is tangent at each point to a principal tangent; it is a line of curvature. Consequently: *The lines of curvature are lines whose geodesic torsion is constantly zero* (a necessary and sufficient condition).

Let us assume that we consider lines Γ of the surface S such that the normals to S along one of these lines Γ engender a developable surface. If M is a point of such a line Γ, then a point of the normal at M is defined by

$$\vec{P} = \vec{M} + \lambda\vec{N} ,$$

and we must seek those conditions in order for the point P to describe a curve whose tangent is MP, and hence of the form $\mu\vec{N}$. We have

$$\frac{d\vec{P}}{ds} = \frac{d\vec{M}}{ds} + \frac{d\lambda}{ds}\vec{N} + \lambda\frac{d\vec{N}}{ds} ,$$

and, on taking account of the third formula (45),

$$\frac{d\vec{P}}{ds} = \vec{t} + \frac{d\lambda}{ds}\vec{N} + \lambda\left(-\frac{\vec{t}}{R}\cos\theta\right) - \lambda\left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{g}$$

or

$$\frac{d\vec{P}}{ds} = \left(1 - \lambda\frac{\cos\theta}{R}\right)\vec{t} - \lambda\left(\frac{1}{T} + \frac{d\theta}{ds}\right)\vec{g} + \frac{d\lambda}{ds}\vec{N} .$$

We must then have

$$\frac{1}{T} + \frac{d\theta}{ds} = 0 , \qquad \lambda = \frac{R}{\cos\theta} ,$$

and this is sufficient.

The curve is defined by the first condition; geodesic torsion zero; this is a line of curvature. The second condition gives the characteristic point of the normal; this is the center of normal curvature. We thus recover the results of nos. 228 and 229.

Likewise, we recover the theorem of Joachimstal in no. 229. If S and S_1 intersect along Γ at a constant angle and if T is the torsion of Γ at M, θ and θ_1 the angles of the osculating plane at M with the tangent planes to S and S_1 at M, then we will have $\theta_1 = \theta + \text{const.}$, hence

$$\frac{1}{T} + \frac{d\theta}{ds} = \frac{1}{T} + \frac{d\theta_1}{ds} ,$$

and if one of the members is zero, the other is also. We indeed deduce the statement of Joachimstal's theorem, and likewise the converse.

Dupin's theorem on triply-orthogonal systems (no. 229) can be obtained without calculations. Taking the notation of no. 229, we initially see that the geodesic torsion of C_3, for example, is the same, whether C_3 is considered on S_1 or on S_2; this results from what has just been said concerning Joachimstal's theorem. Let then, τ_1, τ_2 and τ_3 be the geodesic torsions of C_1, C_2 and C_3 respectively on the surfaces on which they are found. As C_1 and C_2 are orthogonal on S_3, the Bonnet formula (51) shows that their geodesic torsions are opposite; we will therefore have

$$\tau_1 + \tau_2 = 0 \ , \qquad \tau_2 + \tau_3 = 0 \ , \qquad \tau_3 + \tau_1 = 0 \ ,$$

which implies $\tau_1 = \tau_2 = \tau_3 = 0$; C_1, C_2 and C_3 are lines of curvature on their respective surfaces. Q.E.D.

235. Geodesic lines. The Gauss theorem.

In no. 202, we called the geodesics of a surface S given by its ds^2, the lines of the surface which minimize the integral $\int ds$. We have seen that the lines obtained, as defined by the Euler equations, give at least a weak minimum between two neighboring points. If, instead of taking the parametric representation, we take the parameter u as variable, then the extremals, which correspond to

$$\int_A^B \sqrt{E+2Fv'+Gv'^2}\ du \quad ,$$

are given by

$$\frac{\partial}{\partial v}\sqrt{E+2Fv'+Gv'^2} - \frac{d}{du}\frac{F+Gv'}{\sqrt{E+2Fv'+Gv'^2}} = 0 \quad . \tag{58}$$

This is a second order differential equation in v; the coefficient of v'' is never zero. Within a small domain, we can define a sheaf of extremals (hence, of geodesics), for example, orthogonal to a given curve, and such that through each point of the domain there passes one and only one. In the domain in question, we will take these geodesics as coordinate lines, which will be the lines $v = \text{const.}$, and their orthogonal trajectories. Under these conditions we have $F = 0$ and the equation (58) must be satisfied for $v' = 0$, hence we have

$$\frac{\partial\sqrt{E}}{\partial v} = 0, \qquad \sqrt{E} = \chi(u) \quad .$$

When we change the parameter on the geodesic by taking:

$$U = \int \chi(u)\ du \quad ,$$

the ds^2 will take the form:

$$ds^2 = dU^2 + G_1(v,U)\,dv^2 \quad .$$

Hence, *if we assume a part of the surface is referred to a sheaf of geodesics and to their orthogonal trajectories, and if the parameter on the geodesics is the arc of these curves, we have*

$$ds^2 = du^2 + Gdv^2 \quad . \tag{59}$$

The formula (52), which gives the geodesic curvature, then shows that for $u' = 1$, $v' = v'' = 0$, this curvature is zero. *The geodesics have a zero geodesic curvature.* Conversely, the lines whose geodesic curvature is constantly zero, are obtained by equating the second member of the equation (52) to zero. They are given by a second order differential equation when u is taken to be the variable and v as the unknown function. There exists one and only one of these curves which passes through a point and which is tangent to a given direction at this point. It therefore coincides with the geodesic passing through this point. Consequently:

The geodesics are lines with zero geodesic curvature.

<u>Remark</u>. Starting from equation (59), we can verify directly that the lines $v = \text{const.}$ have their osculating plane normal to the surface S at each of their points. Let $\vec{t}$ and $\vec{n}$ be the unit vectors of the tangent and of the principal normal at the point M of the geodesic $v = \text{const.}$ We have $ds = du$, hence:

$$\vec{t} = \vec{M}'_u \; , \qquad \frac{\vec{n}}{R} = \vec{M}''_{uu} \quad .$$

Now $\vec{M}'_u \vec{M}'_v = 0$ gives, on differentiating with respect to u,

$$\frac{\vec{n}}{R}\vec{M}'_v + \vec{t}\,\vec{M}''_{uv} = 0 \quad ,$$

and since

$$\vec{M}'^2_u = 1 \; , \qquad \vec{M}'_u\vec{M}''_{uv} = \vec{t}\,\vec{M}''_{uv} = 0 \quad ,$$

we have:

$$\frac{\vec{n}}{R}\vec{M}'_v = 0 \quad .$$

The vector $\vec{n}$ is hence normal to S.

CONCLUSION

Since the curves with constant zero geodesic curvature are the curves with osculating plane constantly normal to the surface, since $\sin\theta = 0$, $\theta = 0$ or π, and conversely, we see that

The geodesics are the curves whose osculating plane at M *is constantly normal to the surface at the point* M.

At each point M, the geodesic torsion of a geodesic is equal to its torsion ($d\theta = 0$). It follows that:

The geodesic torsion of any curve Γ *at any one of its points is the torsion of the geodesic which is tangent to* Γ *at this point.*

Theorem. *If we consider a regular sheaf of geodesics* Γ *and their orthogonal trajectories* Γ' *in a domain, then the arcs intersected by two curves* Γ' *on the geodesics* Γ *are all equal.*

To say that the sheaf is regular, is to say that through a point of the domain there passes one and only one geodesic of the sheaf. We have a field of extremals (no. 196). Then the ds^2 can be put into the form (59), which we call *the geodesic form of the* ds^2. The arc of the geodesic taken between two curves Γ', $u = u_0$, $u = u$, is

$$\int_{u_0}^{u_1} du = u_1 - u_0 \quad .$$

Q.E.D.

The curves Γ' are *geodesically parallel.*

The condition of transversality is therefore a condition of orthogonality in this case.

Remarks. I. When we join two points AB of S by a geodesic arc taken from a regular sheaf, then the ds^2 can be put into the form (59). The arc AB is equal to $u_1 - u_0$; an arc of a neighboring curve has as its length

$$\int_A^B \sqrt{du^2 + G dv^2} > \int_{u_0}^{u_1} du \qquad (u_0 < u_1) \quad ,$$

since $G > 0$. We have a strict minimum on the arc AB with respect to the curves of the field. This proof is due to Darboux.

II. In particular, we can consider the geodesics passing through a point M_0 and their orthogonal trajectories near to this point. These trajectories are closed curves; they define the curves that play the role of circles. (We can define them by taking on each geodesic, from M_0, an arc of a given length.)

The Theorem of Gauss. *The total curvature*

$$\iint_\Delta \frac{ds}{R_1 R_2}$$

of a simply connected region Δ *of the surface* S, *bounded by a geodesic polygon formed by a finite number* n *of geodesic arcs, is equal to the sum of its interior angles less* $(n-2)\pi$.

In effect, this total curvature is given by

$$2\pi - \left(n\pi - \sum_1^n \alpha_j\right) - \int_\Gamma \frac{ds}{R_g} \quad ,$$

on account of the Bonnet theorem of no. 223, where the α_j are the interior angles of the polygon Γ, and the integral on Γ is zero since the geodesic curvature is zero on the geodesics.

Remarks. The Gauss theorem (1827) long preceded Bonnet's formula (53) (1848). When there exists a closed geodesic line bounding a simply-connected domain, the total curvature of the interior of this line (the integral of the total curvature of the interior points) is 2π.

GEOMETRY ON A SURFACE

If we retain the usual significance of the angles and allow the geodesics to play the role of lines on the Euclidean plane, then we obtain a non-Euclidean geometry on the surface S. The sum of the angles of a triangle (a geodesic triangle, $n = 3$) is

$$\pi + \iint_\Delta \frac{ds}{R_1 R_2} \quad .$$

For a surface with constant positive curvature, $a^2 = R_1 R_2 > 0$ throughout S, we have an analogous geometry to that obtained on a sphere. This is a Riemannian geometry; the sum of the angles of a triangle is greater than π. If the surface has constant negative total curvature, with $-a^2 = R_1 R_2 < 0$ throughout S, then the sum of the angles of a triangle is less than π; we have a Lobatchewskian geometry which was firstly studied by Beltrami; it is similar to the Poincaré geometry (I,166).

236. Isometric surfaces

Given two surfaces S and S_1, every pair of parametric representations of these surfaces establishes a point correspondence between the points of these surfaces. For, if $M(u,v)$ is a point of S defined in terms of two parameters u, v, then the assignment of this representation of S establishes a correspondence between the points of S and the points $m(u,v)$ of the plane of the (u,v).

If $M_1(u_1,v_1)$ is a point of S_1, it corresponds to a point $m_1(u_1,v_1)$ of the plane of the (u_1,v_1). We may assume that the planes (u,v) and (u_1,v_1) are identified $(u_1 = u,\ v_1 = v)$, or establish a correspondence between their points; we will have defined a correspondence between the points M and M_1 of S and S_1. We shall always consider matters in these regions of S and S_1, restricted if necessary, in which these correspondences are bijective. This has always been the case up to now, in order to obtain the correspondence between $M(u,v)$ and $m(u,v)$.

Definition. *We say that S and S_1 are isometric to each other when it is possible to establish a correspondence between their points M and M_1 such that, to every curve arc described on S, there corresponds an arc of the same length described on S_1 and conversely.*

For this to be so, it is necessary and sufficient that, for $M(u,v)$ taken on S and $M_1(u,v)$ to be the corresponding point of S_1, we have

$$\int_{A_1}^{B_1} ds_1 = \int_{A}^{B} ds$$

with

$$ds = \sqrt{Edu^2+2dudv+Gdv^2} \quad , \qquad ds_1 = \sqrt{E_1du^2+2F_1dudv+G_1dv^2} \quad .$$

If we assume A (hence A_1) to be fixed and B (hence B_1) to be variable on a given curve and v a function of u on this curve, then the equality of the integrals (60) implies the equality of their derivatives with respect to u. We must then have:

$$E + 2Fv' + Gv'^2 \equiv E_1 + 2F_1v' + G_1v'^2$$

and v' is arbitrary; it is necessary then that

$$E_1 \equiv E, \qquad F_1 \equiv F, \qquad G_1 \equiv G \ . \tag{61}$$

These necessary conditions are actually sufficient. Consequently:

In order for a surface S_1 to be isometric to a surface S, it is necessary and sufficient to be able to find parametric representations of these surfaces as functions of the same parameters u,v such that the identities in (61) are obtained, i.e. such that the ds^2 are the same.

In short it is necessary and sufficient for the surfaces to have a *same* ds^2.

To see that two surfaces defined by $\vec{M}(u,v)$ and $\vec{M}_1(u_1,v_1)$ are isometric, it is necessary to determine u_1 and v_1 as functions of u and v such that we have

$$E_1du_1^2 + 2F_1du_1dv_1 + G_1dv_1^2 \equiv Edu^2 + 2Fdudv + Gdv^2 \quad ,$$

which yields *three* equations with partial derivatives such that

$$E_1\left(\frac{\partial u_1}{\partial u}\right) + 2F_1\,\frac{\partial u_1}{\partial u}\,\frac{\partial v_1}{\partial u} + G_1\left(\frac{\partial v_1}{\partial u}\right)^2 = E \quad .$$

These equations, which contain only two unknown functions u_1 and v_1, must be compatible, which will not be the case if S and S_1 are arbitrary.

ELEMENTS PRESERVED IN THE ISOMETRY

If S and S_1 are isometric hence have *a* same ds^2 and a same first fundamental form; then all the geometric elements only depending on E,F,G, their derivatives, and the derivatives of u and v with respect to a parameter, and corresponding, will be the same on S and S_1. Consequently:

The angles of two curves at their points of intersection are the same as the angles of their homologues (following no. 222).

The total curvatures of S *and* S_1 *at two homologous points* M,M_1 *are the same* (no. 226).

The geodesic curvature at M *of a curve* Γ *of* S *is equal to the geodesic curvature of the curve homologue* Γ *of* S_1 *at the point* M_1, *the homologue of* M (no. 233). In particular:

The curve on S_1 *corresponding to a geodesic of* S *is a geodesic of* S_1. The geometries on two surfaces isometric to each other, are the same.

Remark. The geodesic torsion of two corresponding curves on two isometric surfaces, is not generally the same at two corresponding points.

A SURFACE ISOMETRIC TO ITSELF

In particular, a surface can be isometric to itself without the correspondence being an identity. A surface of revolution is isometric to itself in an infinity of ways; it suffices to consider a rotation about the axis. Every surface isometric to a surface of revolution will be isometric to itself, where the map depends continuously on one parameter.

The sphere is isometric to itself in an infinity of ways; to a point and a direction at this point, we can assign a given point and a given direction, where the maps in question depend on three parameters. This will be the case for every surface isometric to the sphere. In order for a surface to be isometric to itself in such a way that we may choose the point corresponding to a given point M_0, arbitrarily, it is necessary for the total curvature to be constant. We shall see in no. 237 that this is sufficient.

THE DEFORMATION OF A SURFACE

We say that we can deform a surface S when it is replaced by another surface S_1 isometric to S. This may be realized in material terms when the surface is a thin metallic plate susceptible to curvature without it being bended. *The deformation is then continuous.* We have not demonstrated the possibility of such an operation.

237. Examples. Bour's theorem. Surfaces with total constant curvature.

In the equations (6) of a surface of revolution, we can evidently set $\phi(u) \equiv u$. The surface

$$x = u\cos v\ , \qquad y = u\sin v\ , \qquad z = \psi(u)$$

has then for its ds^2 (no. 222)

$$ds^2 = [1 + \psi'(u)^2]du^2 + u^2dv^2 \quad . \tag{62}$$

This ds^2 can be reduced to the geodesic form: the meridians are geodesics and the parallels are orthogonal trajectories.

Bour's Theorem. *Every helicoid is isometric to a surface of revolution.*

In cylindrical coordinates, a helicoid of which Oz is the axis, has as its equations

$$x = r\cos\omega \ , \qquad y = r\sin\omega \ , \qquad z = a\omega + \chi(r) \ ,$$

where a is a constant. Its ds^2 is

$$ds^2 = [1 + \chi'^2(r)]dr^2 + 2a\chi'(r)drd\omega + (r^2 + a^2)d\omega^2 \quad .$$

We can decompose this expression in such a way as to allow a substitution of the term in r and dr. We have

$$ds^2 = (r^2 + a^2)\left[d\omega + a\,\frac{\chi'(r)}{r^2+a^2}\,dr\right]^2 + \frac{a^2+r^2+r^2\chi'^2(r)}{a^2+r^2}\,dr^2 \quad .$$

If we define u and v by

$$u = \sqrt{a^2+r^2} \ , \qquad v = \omega + \int \frac{a\chi'(r)}{r^2+a^2}\,dr \ ,$$

we obtain

$$ds^2 = u^2dv^2 + \frac{u^2+(u^2-a^2)\chi'^2(\sqrt{u^2-a^2})}{u^2-a^2}\,du^2 \quad .$$

This expression takes the form of (62); in order to establish the identity, it suffices to take

$$1 + \psi'^2(u) = \frac{u^2+(u^2-a^2)\chi'(\sqrt{u^2-a^2})^2}{u^2-a^2} \quad .$$

$\psi(u)$ will be given by a quadrature (the constant of integration corresponds to a translation along Oz, the axis of the surface of revolution; we can take it to be arbitrary).

In particular, *the right helicoid with a direction plane* corresponds to $\chi(r) \equiv 0$, we have $v = \omega$, and

$$\psi'^2 = \frac{a^2}{u^2-a^2}, \qquad \psi = a\int\frac{du}{\sqrt{u^2-a^2}} = a \text{ arg ch } \frac{u}{a},$$

$$u = a \text{ ch } \frac{\psi}{a} .$$

The surface of revolution to which this helicoid is isometric, is generated by the rotation of a pole-chain about its base; *we call it a catenoid.* The rectilinear generators of the helicoid, ω = const., correspond to the meridians v = const. (these are indeed geodesics).

Remark. *The developable helicoid* engendered by the tangents to a circular helix, correspond to

$$\chi(r) = a\left[-\text{arc tg}\frac{\sqrt{r^2-b^2}}{b} + \frac{\sqrt{r^2-b^2}}{b}\right],$$

where b is a constant. Here, we have

$$\psi(u) = \frac{a}{b} u .$$

The surface of revolution is a cone in this case; this is an immediate consequence of the fact that, given the surface is developable, it can only be isometric to a developable surface.

SURFACES WITH CONSTANT TOTAL CURVATURE

Let one such surface and its ds^2 in the geodesic form

$$ds^2 = du^2 + Gdv^2 .$$

We may assume that the sheaf of geodesics utilized is formed by geodesics emanating from a point M_0 obtained for $u = 0$, hence $G(u) \equiv 0$. Furthermore, by changing the parameter v, we can also assume that $\frac{\partial\sqrt{G}}{\partial u}(0,v) \equiv 1$. In effect, by replacing v by $\lambda(v)$, Gdv^2 becomes $G(u,\lambda(v))\lambda'(v)^2dv^2$, and we take

$$\lambda' \frac{\partial\sqrt{G}}{\partial u}(0,\lambda) \equiv 1 .$$

In order to state that the total curvature given by (30), is constant, we must write

$$DD'' - D'^2 = \frac{G^2}{R_1R_2} = kG^2 ,$$

where k is a constant. The value of $DD'' - D'^2$ is given by the Gauss theorem of no. 226. The determinants D'^2 and DD'' here reduce to

$$D'^2 = G\sum x''^2_{uv} - \frac{1}{4}\left(\frac{\partial G}{\partial u}\right)^2$$

$$DD'' = G \sum x''_{u^2} \, x''_{v^2}$$

and we obtain

$$DD'' - D'^2 = \frac{1}{4}\left(\frac{\partial G}{\partial u}\right)^2 - G\tau = \frac{1}{4}\left(\frac{\partial G}{\partial u}\right)^2 - \frac{1}{2} G \frac{\partial^2 G}{\partial u^2} \quad .$$

G is the solution of the differential equation

$$\frac{1}{4} G'^2 - \frac{1}{2} GG'' - kG^2 = 0 \quad ,$$

where the derivatives are tkane with respect to u. This equation is written

$$(\sqrt{G})'' + k \sqrt{G} = 0 \quad ;$$

this a second order equation with constant coefficients in $\sqrt{G}$. We seek the solution for which $G(0,v) = 0$, $\frac{\partial\sqrt{G}}{\partial u}(0,v) = 1$. We obtain

$$G = u^2 \quad , \qquad \text{if} \quad k = 0$$

$$G = \frac{1}{a^2} \sin^2(ua) \quad , \qquad \text{if} \quad k = +a^2$$

$$G = \frac{1}{a^2} \operatorname{sh}^2(au) \quad , \qquad \text{if} \quad k = -a^2 \quad .$$

The ds^2 of a surface with total constant curvature can then be taken in the form

$$du^2 + u^2 dv^2 \; , \qquad du^2 + \frac{1}{a^2} \sin^2(ua) dv^2 \; ,$$

$$du^2 + \frac{1}{a^2} \operatorname{sh}^2(au) dv^2 \quad ,$$

according to whether the total curvature is zero, positive and equal to a^2, or negative and equal to $-a^2$, respectively. The first of these ds^2 is the ds^2 of a plane in polar coordinates (u,v); the second is the ds^2 of a sphere of radius $\frac{1}{a}$. In effect, by substituting u for au, we obtain the expression

$$\frac{1}{a^2}(du^2 + \sin^2 u dv^2)$$

for du^2, which is the form obtained in no. 222 for the ds^2 of a surface of revolution defined by the formulae (6). We can take

$$\phi(u) = \frac{1}{a} \sin u \; , \qquad \psi'(u) = -\frac{1}{a} \sin u \; ,$$

$$\psi(u) = \frac{1}{a} \cos u \; ,$$

which yields the parametric equations of a sphere of radius $\frac{1}{a}$.

On substituting u for au and br for v, where b is a constant less than 1, then the third ds^2 takes the form

$$\frac{1}{a^2}\,(du^2 + b^2 sh^2 u dv^2) \quad ;$$

it corresponds to the surface of revolution defined by the formulae (6), with

$$\phi(u) = \frac{b}{a}\,\mathrm{sh}\,u \;, \qquad \psi'(u) = \frac{1}{a}\sqrt{1-b^2 ch^2 u} \;,$$

$$\psi(\ ) = \frac{1}{a}\int\sqrt{1-b^2 ch^2 u}\;du \quad .$$

The form of the meridian can be easily obtained. This surface Σ' intersects the axis of revolution at the point corresponding to $u = 0$.

It is this point A that corresponds to the point M_0 of the given surface S. But Σ' is also isometric to itself, and the neighborhood of a point B corresponds to the neighborhood of A. Moreover, as the neighborhood of A is isometric to itself under an arbitrary rotation about the axis, we see that a surface with total negative curvature $-a^2$ can be mapped onto Σ', where the point M_0 and a given direction at this point correspond to an arbitrary point B of Σ' and to an arbitrary direction at this point, respectively. We can replace Σ' by another surface of revolution with total negative curvature $-a^2$. Besides the surfaces Σ' (which depend on b), there exists another family depending on one parameter and also the surface Σ engendered by the rotation about Oz of a tractrix whose tangents, taken between the point of contact and Oz, have a length equal to $\frac{1}{a}$.(1) This surface, named a pseudosphere by Darboux, is the most simple of surfaces of revolution with total negative curvature $-a^2$. We can replace Σ' by Σ, we shall say that $\frac{1}{a}$ *is the radius of the pseudosphere.*

We also have the following statements:

Every surface with total curvature zero, is isometric to a plane.
Every surface with constant positive total curvature, is isometric to a sphere.
Every surface with constant negative total curvature, is isometric to a pseudosphere.

Remark. It follows from above that a surface with total constant curvature is isometric to itself continuously, and that the translations on the surface depend on three parameters, which give rise to the geometry of the translations on the surface.

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(1) These results can be easily obtained; in particular, the known property of the center of curvature of the tractrix shows at once that Σ has a constant total curvature.

238. Surfaces isometric to the plane are developable surfaces.

This proposition is contained in the above discussions for nos. 226 and 227. *In order for a surface to be isometric to a plane*, it is necessary for the total curvature to be zero at every point of the surface since the total curvature of a plane is zero. Now the total curvature is

$$\frac{rt-s^2}{(+p^2+q^2)^2}$$

when the surface is taken to be of the solvable form $z = f(x,y)$. It is therefore necessary for $rt - s^2 \equiv 0$ and we know that this implies that the surface is developable.

Conversely, *the total curvature of a developable surface is zero*, hence, the surface is isometric to a plane on account of what we have just proved. Thus:

In order for a surface to be isometric to a plane, it is necessary and sufficient for it to be a developable surface.

Remarks. I. The above statement clearly assumes that the surface is ordinary; the coordinates of a point is expressed in terms of functions of two parameters (u,v) with continuous first and second derivatives. Moreover, we rely on the fact that the total curvature is a geodesic element which is preserved under the isometry; this is a consequence of the Gauss theorem of no. 226 which assumes the existence of the third derivatices. Bonnet gave a direct proof which only involves the second derivatives of the coordinates. But when we no longer assume the existence of these derivatives, the theorem collapses.

Lebesgue proved the existence of non-ruled surfaces which are isometric to the plane.

II. The second part of the proof arises from no. 237. We are going to give a direct proof of this converse, independent of the geodesic ds^2.

Theorem. *When two curves* Λ *and* Λ_1 *have the same expression for the radius of curvatures as a function of the arc, then the developable surfaces engendered by their tangents are isometric in such a way that* Λ *and* Λ_1 *correspond and the rectilinear generators correspond.*

In effect, the surface S corresponding to Λ is defined by equation (7) of no. 221, and its ds^2 is (no. 222)

$$\frac{(v-u)}{R(u)^2}\, du^2 + dv^2 \quad ,$$

where u is the arc of Λ and $R(u)$ is the radius of curvature. We will obtain the same expression for Λ_1, which proves the proposition: to each point P of Γ there corresponds the point P_1 of Λ_1 with the same u;

to M defined by the equality in (7), there corresponds M_1 defined by

$$\vec{M}_1 = \vec{P}_1 + \vec{t}_1(v-u) \quad ,$$

where $\vec{t}_1$ is the unit vector of the tangent to Λ_1 at the point P_1. The generators correspond, along with their orthogonal trajectories.

Corollary. *The surface* S *engendered by the tangents to the points* P *of a space curve* Λ *is isometric to the plane, in the following way. We consider the plane curve* Λ_1 *whose radius of curvature at the point* P_1 *is equal to the radius of curvature of* Λ *at the point* P. *To a point* M *of* S, *situated on the tangent to* Λ *at the point* P, *there corresponds the point* M_1 *of the plane of* Λ_1 *situated on the tangent to* Λ_1 *at the point* P_1 *and such that*

$$\frac{\overrightarrow{M_1P_1}}{\vec{t}_1} = \frac{\overrightarrow{MP}}{\vec{t}} \quad .$$

The curve Λ_1 is defined to within a translation, by its intrinsic equation $R_1 = R(u)$, where $R(u)$ is the radius of curvature of Λ at the point with curvilinear abscissa u. For example, the developable helicoid corresponds to a circle and to its tangents.

Remarks. I. In the case of a cone, P is fixed, and the isometry onto the plane is the operation of development; likewise for the case of a cylinder.

II. In the same way, we see that a space curve is obtained by twisting a plane curve with the same curvature as a function of the arc. This deformation of the curve, along with its tangents, gives the developable surface.

III. THE CONFORMAL REPRESENTATION GEOGRAPHIC CHARTS

239. The conformal representation of a surface onto another surface and onto a plane

A point correspondence between a point M of a surface S and a point M_1 of a surface S_1 is said to be conformal if the angle between two curves Γ_1, Γ_1' of S_1 corresponding to two curves Γ and Γ' of S respectively, is the angle between these curves Γ, Γ', for any such choice.

We can always assume that, for the point $M(u,v)$ of S and the corresponding point M_1 depending on (u,v), the elements of S_1 have been defined in terms of these parameters. Two homologous curves on S and S_1 are defined by taking u and v as functions of a parameter.

The angle between two curves is defined by the formula (13). If E_1, G_1, F_1 are the coefficients of the square of the arc element on S_1,

$$ds_1^2 = E_1 du^2 + 2F_1 dudv + G_1 dv^2 \quad ,$$

then for any $u, v, du, dv, \delta u, \delta v$, we must have

$$\frac{Edu\delta u+F(du\delta v+dv\delta u)+Gdv\delta v}{ds\delta s} = \frac{E_1 du\delta u+F_1(du\delta v+dv\delta u)+G_1 dv\delta v}{ds_1\delta s_1} \quad .$$

On setting $dv = mdu$, $\delta v = m'\delta u$, we obtain:

$$\frac{E+F(m+m')+Gmm'}{\sqrt{E+2Fm+Gm^2}\ \sqrt{E+2Fm'+Gm'^2}} = \frac{E_1+F_1(m+m')+G_1 mm'}{\sqrt{E_1+2F_1 m+G_1 m^2}\ \sqrt{E_1+2F_1 m'+G_1 m'^2}} \quad .$$

This equality must hold for all m and m' (and all u,v). Let us take $m' = 0$. On squaring we have:

$$\frac{(E+Fm)^2}{E(E+2Fm+Gm^2)} = \frac{(E_1+F_1 m)^2}{E_1(E_1+2F_1 m+G_1 m^2)} \quad ,$$

and, on subtracting 1 from each side,

$$\frac{(EG-F^2)m^2}{E(E+2Fm+Gm^2)} = \frac{(E_1G_1-F_1^2)m^2}{E_1(E_1+2F_1 m+G_1 m^2)} \quad .$$

By writing these in the form

$$\frac{E}{EG-F^2}\left(E\,\frac{1}{m^2} + 2F\,\frac{1}{m} + G\right) = \frac{E_1}{E_1G_1-F_1^2}\left(E_1\,\frac{1}{m^2} + 2F_1\,\frac{1}{m} + G_1\right) \quad ,$$

we see that it is necessary to have

$$\frac{E_1}{E} = \frac{F_1}{F} = \frac{G_1}{G} \quad .$$

This necessary condition can be seen to be sufficient. By taking $\Theta(u,v)$ to denote the common value of the ratios in (63), we see that:
In order for the correspondence established between S *and* S_1 *to be conformal, it is necessary and sufficient to have*

$$ds_1^2 \equiv \Theta(u,v)(Edu^2 + 2Fdudv + Gdv^2) \quad ,$$

where the parametric representations are defined by the same parameters u,v, *and the* ds^2 *are proportional.*

We see that this condition implies that, in the neighborhood of two corresponding points, the ratio of the lengths of two small corresponding arcs, only depends on u and v; the transformation will be analogous to a similitude [cf. (I,162)].

If the surface S corresponds conformally to S_1 and if the surface S_1 corresponds conformally to S_2, then the correspondence between S and S_2 is also conformal. Therefore, in order to study the conformal correspondences, we reduce matters to the case where S_1 is a given particular surface. We will take a plane. When S and S_1 correspond to each other conformally, we say that S is conformally represented on S_1 (or conversely, S_1 on S). *Matters are reduced to studying the conformal representations of a given surface onto a plan.*

Theorem. *Every analytic surface* S *can be conformally represented onto a plane.*

Let us assume the surface to be analytic. The functions E,F,G will be analytic functions of u and v and it will be possible for us to assign complex values to u and v. We can decompose ds^2 into a product of factors

$$ds^2 = E\left(du + \frac{F+i\sqrt{EG-F^2}\,dv}{E}\right)\left(du + \frac{F-i\sqrt{EG-F^2}\,dv}{E}\right) ,$$
$$i = \sqrt{-1} .$$

Following the theory of analytic differential equations (no. 46), the expression

$$du + \frac{F+i\sqrt{EG-F^2}\,dv}{E}$$

admits an integrating factor $\Omega(u,v)$ which is an analytic function of u and v. The second differential expression appearing in the ds^2 admits the conjugate imaginary integrating factor since it is deduced from the first on changing i to $-i$. We obtain

$$\Omega(u,v)\left[du + \frac{F+i\sqrt{EG-F^2}\,dv}{E}\right] \equiv dU(u,v)$$

$$\bar{\Omega}(u,v)\left[du + \frac{F-i\sqrt{EG-F^2}\,dv}{E}\right] \equiv d\bar{U}(u,v) ,$$

where the functions U and $\bar{U}$, Ω and $\bar{\Omega}$ are imaginary conjugates (u,v real). It follows that

$$|\Omega|^2 ds^2 = E\,dU\,d\bar{U} , \qquad \frac{|\Omega|^2}{E}\,ds^2 = dU\,d\bar{U} .$$

Now

$$U = X(u,v) + iY(u,v)$$
$$dU\,d\bar{U} = dX^2 + dY^2 ,$$

such that

$$dX^2 + dY^2 = \frac{|\Omega|^2}{E}\, ds^2 \quad ,$$

shows that S is conformally represented onto the plane OXY (rectangular axes), since the square of the line element of the plane is $dX^2 + dY^2$.

Remarks. I. It suffices to put the ds^2 of any surface in the form

$$\Omega(u,v)/(du^2 + dv^2) \quad , \tag{64}$$

in order for it to define a conformal representation of S onto the plane with rectangular coordinates (u,v).

II. When we have made a conformal representation of a surface S onto a plane, we can deduce all others by considering the conformal representations of the plane onto itself. These conformal representations are given by analytic transformations and by these transformations composed with a-symmetry (I,162).

III. It suffices to put the ds^2 into the form

$$\Omega(u,v)[\omega(u)du^2 + \omega_1(v)dv^2] \quad , \tag{65}$$

in order for it to be readily stated in the form (64) by setting

$$\int\sqrt{\omega(u)}\,du = dU \ , \qquad \int\sqrt{\omega_1(v)}\,dv = dV \quad .$$

Under these forms, (64) or (65), the system of coordinate curves is orthogonal and *isothermal* (we may also say *isometric*). In (64), for any u_0 and v_0, the curves $u = u_0$, $u = u_0 + \delta$, $v = v_0$, $v = v_0 + \delta$, where δ is given to be infinitely small, form a small square on the surface.

240. Examples. Geographic charts.

The ds^2 of the ellipsoid given in no. 222 can be explicated in the form

$$ds^2 = \frac{1}{4}(v-u)\left[\frac{-udu^2}{(a^2+u)(b^2+u)(c^2+u)} + \frac{vdv^2}{(a^2+v)(b^2+v)(c^2+v)}\right] .$$

The system of curves obtained by intersecting the ellipsoid with homofocal hyperbolloids is an isothermal system. By setting

$$X = \int\left[\frac{-u}{(a^2+u)(b^2+u)(c^2+u)}\right]^{1/2} du \quad ,$$

$$Y = \int\left[\frac{v}{(a^2+v)(b^2+v)(c^2+v)}\right]^{1/2} dv \quad ,$$

we achieve the conformal representation of the ellipsoid onto the plane, where the system in question gives the parallels to the axes. The eigth of the ellipsoid is represented onto a rectangle.

GEOGRAPHIC CHARTS

They correspond to the representation of the sphere onto a plane. The conformal charts are deduced from the stereographic projection (which is an inversion, and hence preserves the angles) by an analytic transformation. The Mercator projection provides a conformal chart in which the meridians and parallels are mapped onto (straight) lines. They are hence obtained from the stereographic projection in which the meridians have as their images the concurrent lines, and the parallels, the circles centered at this point of concurrence, via a logarithmic transformation (I,172).

Let us determine the Mercator projection explicitly by taking the sphere to be in the form

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta .$$

The ds^2 is $R^2(d\theta^2 + \sin^2\theta d\phi^2)$, and on writing

$$R^2 \sin^2\theta \left[\frac{d\theta^2}{\sin^2\theta} + d\phi^2\right] ,$$

we see that we can obtain a chart in which the meridians $\phi = \text{const.}$, and the parallels $\theta = \text{const.}$, determine lines, by taking the point

$$X = k\int \frac{d\theta}{\sin\theta} = k \log \operatorname{tg} \frac{\theta}{2} , \qquad Y = k\phi$$

to be the point corresponding to the point θ, ϕ, where k is a constant. The preceding result is recovered by these direct means.

NON-CONFORMAL GEOGRAPHIC CHARTS

We have considered charts where the areas of the homologous domains are in a fixed ratio. If E,F,G are the coefficients of the first fundamental form of a surface S and if at the point M of S we assign the point $M_1(X,Y)$ of the plane OXY, where X,Y are functions of u and v, then the ratio of the areas will be preserved if

$$\iint dXdY = \iint \frac{D(X,Y)}{D(u,v)}\, dudv = K\iint \sqrt{EG-F^2}\, dudv ,$$

where K is a constant. It is necessary and sufficient to have

$$\frac{D(X,Y)}{D(u,v)} = K\sqrt{EG-F^2} ;$$

we can take X to be arbitrary and Y will be given by a partial differential equation whose solution will contain an arbitrary function.

For example, for a surface of revolution S given by the formulae in (6), we have

$$EG - F^2 = \phi^2(\phi'^2 + \psi'^2) \quad .$$

Let us assume that we require the parallels $u = \text{const.}$, to be represented by circles with center 0 in the plane of the X,Y, where areas are preserved. By taking the polar coordinates R, Θ in the plane of the X,Y, we will obtain the condition

$$R \frac{D(R,\Theta)}{D(u,v)} = K\phi \sqrt{\phi'^2 + \psi'^2} \quad ,$$

where R depends only on u. We will have

$$R \frac{\partial R}{\partial u} \frac{\partial \Theta}{\partial v} = K\phi \sqrt{\phi'^2 + \psi'^2} \quad ;$$

Θ will be a linear function of v, with coefficients depending on u. If we take $\Theta = kv$, $k = \text{const.}$, R will be determined by a quadrature. For a sphere, we will have $\phi = a \sin u$, $\psi = a \cos u$, hence, for example

$$R^2 = 2 \frac{K}{k} a^2(1 - \cos u) \quad .$$

Chapter XIV

THE THEORY OF SURFACES -- APPLICATIONS

In this second chapter on the theory of surfaces, we shall commence by discussing the applications of the general theorems to the study of asymptotic and curvature lines, in particular, we present a brief discussion of the cycloids of Dupin and moulding surfaces; this is followed by a study of ruled, non-developable surfaces.

The properties of families of two parameter curves, known as congruences in the Plücker terminology, which will subsequently be stated, have been related to the theroy of singular integrals of differential systems. The condition for a line congruence to be a congruence of normals, was known to Dupin and Malus (1817), but this was completed by Bertrand as late as 1844. To this exposition of the properties of congruences, we have included a discussion on ruled surfaces belonging to a complex.

The study of minimal surfaces which then follows, was taken up by Lagrange (1760), Meusnier (1776), Monge (1784) and Legendre. An independent problem to that posed by Lagrange, which consisted of passing a surface of minimal area through a given contour, was solved in later years by Bonnet (1853), Riemann (1861), Weierstrass (1866), Schwarz (1867) and Lie (1877). We shall confine ourselves to stating the general formulae defining a minimal surface and discussing the simplest examples of such surfaces, as given by Meusnier, Catalan and Enneper (1964).

For the solutions of the main problem as given by Garnier and Douglas, we refer to the papers of these authors (see no. 260). Finally, we state the Codazzi formulae (1859) and comment upon the problem of the determination of a surface by its two fundamental quadratic forms.

I. ASYMPTOTIC LINES AND LINES OF CURVATURE OF SURFACES

241. Asymptotic lines of surfaces

1. The surfaces of the form

$$xf\left(\frac{y}{x}\right) = F(z) \ ,$$

studied by Janet, are written, on setting $y = ux$, as

$$x = \frac{F(z)}{f(u)} \ , \qquad y = \frac{uF(2)}{f(u)} \ .$$

We have

$$pF'(z) = f(u) - uf'(u) , \qquad qF'(z) = f'(u) .$$

The asymptotic lines are given (no. 227) by $dpdx + dqdy = 0$:

$$\frac{F''(z)}{F(z)} dz^2 - \frac{f''(u)}{f(u)} du^2 = 0 .$$

Matters are reduced to two quadratures

$$\int\sqrt{\frac{F''(z)}{F(z)}}\, dz = \pm\int\sqrt{\frac{f''(u)}{f(u)}}\, du .$$

Entering into this case are the surfaces with equation $x^m y^n z^p = 1$, where m, n, p are constants (we can always reduce matters to $p = 1$), a case in which the quadratures can be quite easily effected, and also the tetrahedral surfaces defined by

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^n + \left(\frac{z}{c}\right)^m = 1 ,$$

where a, b, c are constants. A homographic transformation reduces it to the canonical form $x^m + y^m = 1 - z^m$. It can be seen that the integrations are possible. When m is rational, the asymptotic lines of the tetrahedral surface are algebraic.

Remarks. I. The definition of the asymptotic lines in terms of the property of the osculating plane is manifestly projective; *the homographic transformations preserve the asymptotic lines*, signifying that to an asymptotic line of a surface there corresponds an asymptotic line on the transformed surface. More generally, two conjugate systems Γ and Γ' on S give two conjugate systems on the transformed surface, since the developable surface circumscriptive to S along Γ, gives a developable surface circumscriptive to S_1 along the transformation Γ_1 of Γ, where the generators of these developables correspond to each other under the transformation.

II. In a similar way, a transformation by duality induces a correspondence between two conjugate directions of a surface S and two conjugate directions of the transformed surface S_1, since to a line Γ of S there corresponds a developable surface γ circumscriptive to S_1, and to a tangent of Γ there corresponds a generator of γ.

Again, there is a preservation of the set of asymptotic lines. This occurs, in particular, in the case of a Legendre transformation (I, 134).

III. The conoids of the form $z = f(\frac{y}{x})$ belong to the general category of ruled surfaces (rectilinear generators $z = \text{const.}$), and the discussion of these surfaces shows that the asymptotic lines must be obtained by a quadrature. Let us determine these directly. By setting $y = vx$, we have

$$x = u , \qquad y = uv , \qquad z = f(v) .$$

The asymptotic lines are given by $(\vec{M}'_u, \vec{M}'_v, d^2\vec{M}) = 0$:

$$\begin{vmatrix} 1 & v & 0 \\ 0 & u & f'(v) \\ 0 & 2dudv & f''(v)dv^2 \end{vmatrix} = 0 \ .$$

We therefore have $dv = 0$ which yields the rectilinear generators $v = \text{const.}$, and

$$\frac{2du}{u} = \frac{f''(v)dv}{f'(v)} \ ,$$

giving $u^2 = Cf'(v)$. The projections of the asymptotic lines of the second system onto the plane Oxy are defined by

$$x^2 = Cf'\left(\frac{y}{x}\right) \ ,$$

where C is an arbitrary constant.

IV. For a helicoid

$$x = u \cos v \ , \qquad y = u \sin v \ , \qquad z = av + \psi(u) \ ,$$

where a is a constant, a similar calculation to the one before gives the equation

$$u\psi''(u)du^2 - 2adudv + \psi'(u)u^2dv^2 = 0 \ ,$$

which can be integrated by quadratures. In the particular case of the surfaces of revolution, $a = 0$, we have

$$v = \pm\int\sqrt{-\frac{\psi''}{u\psi'}}\, du + \text{const.}$$

In the case of a torus, we recover an elliptic integral.

242. <u>Lines of curvature. The formulae of Rodrigues. Inversion preserving the lines of curvature</u>.

In no. 227, we gave the equation for the lines of curvature (formula 37). But we can obtain this equation directly by taking, along such a line, the normal to an envelope (no. 228), as was the case in no. 234 by utilizing the geodesic curvature. We proceed directly as follows. $\vec{M}$ is taken to be the vector which defines the surface, $\vec{N}$ the unit vector of the normal; a point P of the normal, defined by $\vec{P} = \vec{M} + \lambda\vec{N}$, must describe a curve tangent to MP. Hence

$$d\vec{P} = d\vec{M} + \lambda d\vec{N} + d\lambda\vec{N} = k\vec{N} \ .$$

As the vectors $d\vec{M}$ and $d\vec{N}$ are normal to $\vec{N}$, it is necessary and sufficient that

$$d\vec{M} + \lambda d\vec{N} = 0 \ .$$

THIS RELATIONSHIP IS DUE TO RODRIGUES

It indicates the direction of principal tangents and the radii of principal curvature which will be the values of λ. On denoting by R this radius of curvature measured on $\vec{N}$, we must have

$$d\vec{M} + Rd\vec{N} = 0 \quad . \tag{1}$$

More explicitly, we have

$$\vec{M}'_u du + \vec{M}'_v dv + R(\vec{N}'_u du + \vec{N}'_v dv) = 0$$

and, on multiplying by $\vec{M}'_u$ then $\vec{M}'_v$, we obtain

$$Edu + Fdv - R(\mathfrak{L}du + \mathfrak{M}dv) = 0$$

$$Fdu + Gdv - R(\mathfrak{M}du + \mathfrak{N}dv) = 0 \quad .$$

On eliminating R, we recover equation (37) of no. 227; on eliminating dv/du, we obtain the equation of the radii of principal curvature.

The relation (1) of Rodrigues can also be stated by projecting onto the axes

$$dx + Rd\alpha = 0 \,, \qquad dy + Rd\beta = 0 \,, \qquad dz + Rd\gamma = 0 \,, \tag{2}$$

where α, β, γ are the projections of $\vec{N}$ onto the axes. *The formulae in* (2) *are the formulae of Rodrigues (1816).*

THE CASE OF A SURFACE $z = f(x,y)$

The normal has as its equations

$$X = -pZ + (x+p) \,, \qquad Y = -qZ + (y+qz)$$

(in the notation of Monge). It engenders a developable surface when

$$\frac{d(x+pz)}{dp} = \frac{d(y+qz)}{dq} \tag{3}$$

(no. 61), which states that the contribution of the characteristic point provided by the derived equations

$$Z = \frac{d(x+pz)}{dp} \,, \qquad Z = \frac{d(y+qz)}{dq} \quad ,$$

is the same. Equation (3) can be simplified and reduces to

$$\frac{dx+pdz}{dp} = \frac{dy+qdz}{dq} \quad . \tag{4}$$

From this, we deduce the coordinates of the corresponding center of curvature, where $Z - z$ is the common value of the ratios in (4); the radius of curvature will be

$$R = \sqrt{1+p^2+q^2}\ (Z-z) .$$

<u>Example</u>. Consider an ellipsoid with respect to its axes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 ;$$

we have

$$c^2x + a^2pz = 0 , \qquad c^2y + b^2qz = 0 . \tag{5}$$

Equation (4) therefore becomes

$$a^2(b^2 - c^2)dpdy + b^2(c^2 - a^2)dqdx = 0 .$$

The equations in (5) yield

$$a^2z^2dp = c^2(xdz - zdx) , \qquad b^2z^2dq = c^2(ydz - zdy)$$

and the equation of the lines of curvature becomes

$$(b^2 - c^2)(xdz - zdx)dy + (c^2 - a^2)(ydz - zdy)dx = 0 .$$

On multiplying by z and taking account of the equation of the ellipsoid which also gives zdz, we obtain the equation of the lines of curvature onto the plane Oxy,

$$a^2(b^2 - c^2)xyy'^2 + y'[a^2(c^2 - b^2)y^2 + b^2(a^2 - c^2)x^2 - a^2b^2(a^2 - b^2)] - b^2(a^2 - c^2)xy = 0 .$$

This is the equation as studied by Monge, and was integrated in no. 73.

PARALLEL SURFACES

Given a surface S, let us take on each normal MN, starting from M, a vector $\vec{MP} = k\vec{N}$, where k is a constant. We obtain a surface $S(k)$ which we call a surface parallel to S. If Γ is a line of curvature of S, and Λ the edge of regression of the normality formed by the normals to S along Γ, then Γ is an involute of Λ. The points of $S(k)$ situated on this normality are points P such that $\vec{PM} = k\vec{N}$, they are on another involute of Λ, $\Gamma(k)$ say. Similarly, the normality formed by the normals of S along a line of curvature Γ' of the second system intersects $S(k)$ along a curve $\Gamma'(k)$ which is an involute of the edge of regression Λ' of this normality. Through the point P there passes a curve $\Gamma(k)$ and a curve $\Gamma'(k)$ these being normals to MP. It follows that:

The normal MP *of* S *is also normal to* $S(k)$ *at the point* P*; the normalities of* S *and* $S(k)$ *are the same. A surface* S *and its parallel surfaces*

$S(k)$, *where* k *is arbitrary, form, along with the normalities of* S, *a triply-orthogonal system.*

Theorem. *Inversion preserves the lines of curvature. Otherwise said, if we transform by inversion a surface* S *to a surface* S_1, *then the lines of curvature of* S_1 *are the transformations, under this inversion, of the lines of curvature of* S.

In effect, let us regard S as belonging to a triply-orthogonal system formed by the surfaces $S(k)$ parallel to S and the two systems Σ and Σ' of normalities of S. The lines of curvature of S are the sections of S through the surfaces Σ and Σ'. In the inversion, which preserves the angles, the surfaces $S(k)$, Σ and Σ' are transformed into $s(k)$, σ and σ' which again constitute a triply-orthogonal system. The lines of curvature of s are the intersections of s and of the surfaces by virtue of Dupin's theorem (no. 229); they are indeed the inverses of the intersections of S with Σ and Σ'.

Remark. It is possible to give a large variety of proofs of the above theorem. For example, we might remark that, if S is a developable surface, then its lines of curvature are its rectilinear generators G and their orthogonal trajectories Γ and that the generators G are the characteristics of the tangent planes π. An inversion transforms S into a surface s, an envelope of spheres Σ inverse to the planes π. The characteristics of these spheres are the circles g inverse to the generators G; these are the lines of curvature of s, hence also are their orthogonal trajectories γ which are the inverse curves of Γ. The theorem is therefore true for a developable surface. If S is arbitrary, it is orthogonal to the normality corresponding to a line of curvature Γ. The transformation of S under inversion, is orthogonal along the transformation γ of Γ, to the inverse surface of the normality, where γ is a line of curvature on this last surface, as it is also on the transformation of S by virtue of Joachimstal's theorem.

Another proof will arise out of a later discussion concerning the surface envelopes of spheres.

243. Surfaces with lines of circular curvature. Envelopes of spheres. The cycloids of Dupin.

Let us consider, firstly, a family of spheres Σ depending on one parameter. Following the general theory (no. 59, example II), the characteristics are circles. If A is taken to be the center of the sphere Σ, R its radius, then the characteristics circle is a plane perpendicular to the tangent to A at the locus of the center.

On confining our attention to the real numbers, as is the case here, we find that three cases are possible.

First case. The characteristic circle of C is a true circle, with real radius nonzero, for the points A belonging to an arc Γ of the locus of the center. C is a line of curvature of the envelope surface S, and the normals to S along C form a cone of revolution with vertex A whose axis is the tangent to the curve Γ at the point A. The sheet of the focal surface of the normals to S corresponding to the normalities relative to the circles C, is reduced to the curve Γ. The second system of lines of curvature of S is formed by the orthogonal trajectories of the circles C.

Second case. The characteristic circle C reduces to a point for any A on an arc Γ of the locus of the centers A. The real part of the envelope reduces to the curve described by this characteristic point B. This curve Λ admits Γ as a development.

This situation arises when we consider a surface S', a line of curvature Λ' of S' and the edge of regression Γ' of the normality corresponding to Λ': the spheres having their centers on Γ' and tangent to S' along Λ' admit Λ' as the only real envelope. The system formed by S' and these spheres Σ' are transformed into an analogous system, by inversion, which provides another proof of the theorem on the inversion of the lines of curvature.

Third case. The circles C are not real; the spheres E do not have an envelope from the real-geometric point of view.

CONVERSES. A SURFACE OF WHICH ONE SHEET OF THE SURFACE OF THE CENTERS REDUCES TO A CURVE OR SURFACE OF WHICH ONE SYSTEM OF LINES OF CURVATURE IS CIRCULAR

Let us consider a surface S of which one of the systems of lines of curvature is formed by circles C. Following the geometric definition of the lines of curvature, the normals to S along C form a developable surface, and therefore are tangent to a development of C, a development which reduces to a point. The normality relative to C is a cone of revolution with vertex A; the sphere Σ of center A passing through C is tangent to S along C. The surface S is an envelope of spheres Σ; the sheet of the surface of the centers relative to the circles C is the locus Γ of the point A.

Let us consider a surface S and establish, *a priori*, the hypothesis that one of the sheets of the surface of the centers, relative to one of the systems of lines of curvature, C, reduces to a curve Γ. The normals to S along one of the lines C intersect Γ. They intersect it at the same point A. For, on the contrary, they would be tangent to Γ along an arc Σ, and when C is displaced, this arc Γ would engender a surface. The normals to

S along C are therefore concurrent at a point A of Γ; the curve C admits a development at the point A. C is a spherical curve described on a sphere Σ of center A, which is tangent to S along C. S is an envelope of spheres with one parameter.

CANAL SURFACE

As we mentioned in no. 59, this is the envelope S of spheres of constant radius whose centers A describe a curve Γ. The characteristic circle C is in the normal plane to Γ at A; its radius is constant. All lines of curvature C are therefore equal circles. The lines of curvature of the second system are the orthogonal trajectories of these circles C. If Γ' is one of these lines, then the normals to S along Γ' intersect Γ orthogonally and Γ' is an involute of Γ.

Canal surfaces enter into the case of moulding surfaces which are to be discussed in no. 244.

Remark. When the radius R of Σ is a function of the arc s of Γ, then the square of the radius of C is $R^2(1 - R^2)$ (no. 59); from this we deduce all cases where this radius is constant.

CYCLOIDS OF DUPIN

A surface S whose surface of the centers is reduced to two curves has, following the above discussion, two systems of lines of curvature and vice-versa. It has two different kinds of envelopes of spheres depending on one parameter; let Σ and Σ' be these two families. If Σ_1, Σ_2, Σ_3 are three of the spheres Σ, then every sphere Σ' is tangent to these three spheres. The curve Γ', the locus of the centers of the spheres Σ' is a conic. In effect, the locus of the centers of the spheres tangent to Σ_1 and Σ_2 is formed by two quadrics; Γ' is on one of these quadrics Q_{12}; this is a quadric engendered by the rotation of a conic having as its foci the centers of Σ_1 and Σ_2. Likewise Γ' belongs to a quadric Q_{13}. A point P of Q_{12} can be defined in terms of the center A_1 of Σ_1 and the corresponding direction plane π_{12}. Taking p_{12} to be the projection of P onto π_{12}, we have $|\overrightarrow{Pp_{12}}| = k_{12}|\overrightarrow{PA_1}|$, where k_{12} is a constant. Likewise, we have $|\overrightarrow{Pp_{13}}| = k_{13}|\overrightarrow{PA_1}|$, which shows that $|\overrightarrow{Pp_{12}}|k_{13} = k_{12}|\overrightarrow{Pp_{13}}|$, and P is in a plane. Γ' is the intersection of Q_{12} and a plane; this a conic. Similarly, the locus of the centers of the spheres Σ is a conic Γ. The normals of S are the cones of revolution whose vertices are on Γ and which contain Γ' and the cones of revolution whose vertices are on Γ' and which pass through Γ. *When* Γ *reduces to a line,* Γ' *is therefore a circle admitting* Γ *as an axis.* S is the envelope of spheres centered on Γ' and tangent to a sphere centered on Γ; this is *a torus*. We see this as a true torus when the characteristic circles of the spheres Σ' do not intersect Γ,

and it is regarded as a surface engendered by circles equally tangent to a fixed line at a fixed point, or by equal circles passing through two points in the other case. Putting this case aside, Γ and Γ' are true conics: Γ is an ellipse and Γ' a hyperbola situated in two rectangular planes, with each of the curves passing through the foci of the other. Or Γ and Γ' are two parabolas situated in two rectangular planes, each passing through the focus of the other. These are focal conics.

These surfaces S are the cycloids of Dupin. The inverse of a cycloid of Dupin, is generally a cycloid of Dupin. However, when the spheres Σ_1, Σ_2, Σ_3 for example, have a real point in common, then all the spheres Σ pass through this point 0. Now when we effect an inversion of which 0 is the pole, then the characteristic circles of the spheres Σ all of which pass through 0 and through a second point 0', or which are tangent to the same line at 0, have as their inverses, the lines passing through a fixed point at a finite or infinite distance, and the inverse surface is a cone or a cylinder of revolution. *These are the reduced figures corresponding to the cycloids of Dupin having two conical real double points or a real singular point of the second kind envisaged.* Generally, by inverting with respect to a point 0 of the orthogonal circle common to Σ_1, Σ_2, Σ_3 we obtain a torus, since the spheres corresponding to these three spheres will have their centers aligned. This torus will be either a true torus or a torus of the type described above. *The true torus will be the reduced figure corresponding to the cycloids of Dupin without real singular points.*

The reduced figures can serve to provide a study of all the properties which remain invariant under an inversion. For example, given a true torus, we refer to the circle orthogonal to all of the spheres inscribed along the meridinal circles as the principal circle of the torus. There exists an infinity of tori which admit a given axis Γ and a given principal circle Γ' (Γ is the axis of Γ'). These tori form a triply-orthogonal system with the planes passing through Γ and the spheres passing through Γ'. Such a system is preserved by an inversion. By inversion, we thus obtain all of the cycloids of Dupin without double points. We obtain analogous systems for the two other types of cycloids of Dupin. For example, we might consider the cones of revolution having a given vertex and a given axis, where the planes pass through the axis and the spheres have their center at the vertex.

All of the definitions and properties of the torus only involving circles spheres and angles, will extend to the cycloids. The planes on spheres bitangent to a cycloid of Dupin, intersect this cycloid along two circles corresponding to the Villarceau circles of the torus. We thus have two new families of circles on a cycloid without double points. Two circles belonging to one or another of these families are said to be 'paratactic' circles; every sphere passing through one, intersects the other at a constant angle. [See *Geométrie* by Hadamard (3rd edition).]

Remark. The inverse surface of a cone or cylinder of the second degree is a surface S which is an envelope of spheres Σ corresponding to the tangent planes of the cone or of the cylinder. These are spheres passing through two fixed points or are tangent at a point to a line. The lines of curvature of the second system of S correspond to nonrectilinear lines of curvature of the cone or of the cylinder which are spherical in the case of the cone and planar in the case of the cylinder. These are, therefore, spherical curves. On S we find two families of circles corresponding to the circles of the cone or of the cylinder.

244. Moulding Surfaces

Consider a skew curve Γ described by a point M; let s be the arc computed from the point M_0, and let $\vec{t}$, $\vec{n}$, $\vec{b}$, R and T be the usual quantities. When the developable surface engendered by the tangents of Γ is mapped onto a plane π, the curve Γ is mapped onto a curve γ; M_0 is mapped to m_0, M to m, the arc m_0 is equal to s, the vectors $\vec{t}$ and $\vec{n}$ happen to coincide with the unit vectors $\vec{\tau}$ and $\vec{\mu}$ of γ at the point m, and the radius of curvature of γ at m is equal to R. Let us consider a curve λ of the plane π invariably tied to γ. Let us map the plane π onto the osculating plane to Γ at the point M in such a way that the vectors $\vec{\tau}$ and $\vec{\mu}$ of γ coincide with $\vec{t}$ and $\vec{n}$; the curve λ will take a position Λ. When M describes Γ, the curve Λ describes a surface S which is called a *moulding surface*. We also say that the curve γ is forced to roll, without sliding, on Γ, in such a way that its plane constantly coincides with the osculating plane of Γ.

Theorem. *Each point P of Λ describes a curve whose tangent is constantly perpendicular to the osculating plane.*

In effect, let p be the point of γ corresponding to P. We have

$$\vec{P} = \vec{M} + k\vec{t} + k'\vec{n}\ , \qquad \vec{p} = \vec{m} + \vec{k}\tau + \vec{k}'\vec{\mu}\ .$$

On differentiating with respect to s, we obtain

$$\frac{d\vec{P}}{ds} = \left(1 + \frac{dk}{ds} - \frac{k'}{R}\right)\vec{t} + \left(\frac{k}{R} + \frac{dk'}{ds}\right)\vec{n} + \frac{\vec{b}k'}{T}$$

and

$$0 = \left(1 + \frac{dk}{ds} - \frac{k'}{R}\right)\vec{\tau} + \left(\frac{k}{R} + \frac{dk'}{ds}\right)\vec{n}\ ,$$

hence

$$\frac{d\vec{P}}{ds} = \frac{k'}{T}\ \vec{b}\ . \qquad \text{Q.E.D.}$$

<u>Remark</u>. Conversely, every point P of the osculating plane which describes a curve whose tangents are constantly normal to the plane, is invariably tied to γ.

<u>Corollary</u>. *The lines of curvature of the moulding surface* S *are the generating curves* Λ *and the trajectories of their points* P.

In effect, the plane curve Λ whose plane intersects S at a right angle is a line of curvature, and the trajectories of the points P are the orthogonal trajectories of these curves.

<u>Remark</u>. The curves Λ are the geodesics of S since their plane is orthogonal to S.

CONVERSE

If the lines of curvature of a system belonging to a surface S, *are geodesics, then* S *is a moulding surface.* Let Λ be one of the lines of curvature which are the geodesics. The normals to S along Λ are the principal normals of Λ; these normals having an envelope, Λ is a plane curve (no. 213).

The plane of Λ is orthogonal to S and for its envelope it has a developable surface; let Γ be the edge of regression of this surface. If P is a point of Λ which describes a line of curvature of the second system, when Λ is displaced, then the trajectory of the point P situated in the osculating plane to Γ is constantly normal to this osculating plane.

Following the remark to the above theorem, P is invariably tied to γ. Λ is therefore invariably tied to γ, and the surface S is a moulding surface.

LIMITING CASES

If the curve Γ is reduced to a point, then the developable surface attached to Γ reduces to a cone having this point A as its vertex; γ reduces to a point a, and the plane π rolls along the cone without sliding. The lines of curvature of the second system, the trajectories of the points of Λ, are spherical curves.

When the point A is at infinity. the developable surface is a cylinder, and the plane π rolls without sliding on this cylinder. The lines of curvature of the second system are plane curves, which are the involutes of the line segments of this cylinder. When the cylinder is reduced to a line, then the moulding surface is reduced to a surface of revolution.

Remark. A surface S for which the lines of curvature of one of the systems, are planar and are situated in the parallel planes π' between them, is a moulding surface. Since, following Koenigs theorem (no. 228), the lines of curvature of the second system are the contact curves of the cylinders circumscribed by S and parallel to the various directions of the planes π', we deduce that they are planar and there are, in fact, geodesics; indeed, this is the case where the planes π of these curves have a cylinder as their envelope.

245. Various problems relating to curves on a surface.

I. Trajectories of curves of a surface at a constant angle V

We proceed as in no. 222 in the case of orthogonal trajectories. If $M(u,v)$ is a point of the surface S and

$$h\left(u,v,\frac{\delta v}{\delta u}\right) = 0$$

the differential equation of the curves from which the trajectories at the angle V are determined, then on eliminating $\frac{\delta v}{\delta u}$ from this equation, the condition

$$\left[E + F\left(\frac{\delta v}{\delta u} + \frac{dv}{du}\right) + G\,\frac{dv}{du}\,\frac{\delta v}{\delta u}\right]^2$$
$$= \cos^2 V\left[E + 2F\,\frac{\delta v}{\delta u} + G\left(\frac{\delta v}{\delta u}\right)^2\right]\left[E + 2F\,\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2\right] ,$$

gives the differential equation of the curves in question.

In particular, if the given curves are one of a family of coordinate curves, e.g. $u = \text{const.}$, then the equation simplifies to

$$\left(F + G\,\frac{dv}{du}\right)^2 = \cos^2 VG\left[E + 2F\,\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2\right] .$$

Example. If $z = f(x,y)$ is the equation of S, then the lines intersecting the curves $z = \text{const.}$ at angle V correspond to

$$p\delta x + q\delta y = 0 ,$$

and we obtain

$$\left(q - p\,\frac{dy}{dx}\right)^2 = \cos^2 V\left[(1+p^2) + 2pq\,\frac{dy}{dx} + (1+q^2)\left(\frac{dy}{dx}\right)^2\right](p^2+q^2) .$$

This lends itself to a straightforward calculation, in which the direction parameters are

$$q,\ -p,\ 0; \quad dx,\ dy,\ pdx + qdy .$$

For $V = \frac{\pi}{2}$, we obtain *the gradient lines* of S, where Oxy is assumed to be horizontal.

II. The condition for the lines $u = \text{const.}$, $v = \text{const.}$, to be conjugate

We have seen (no. 227) that this condition is $D' \equiv 0$. Now we know that for a determinant to be zero, it is necessary and sufficient that there should exist an equal linear and homogeneous relation between the elements of the lines or columns. The coordinates x,y,z of a point of the surface, as functions of u,v, will therefore happen to satisfy a similar relation of the form

$$\frac{\partial^2 w}{\partial u \partial v} + A(u,v)\frac{\partial w}{\partial u} + B(u,v)\frac{\partial w}{\partial v} \equiv 0 , \qquad (w = x,y,z) \quad .$$

In particular, if we take $A(u,v) \equiv B(u,v) \equiv 0$, we obtain the solutions

$$x = \phi(u) + \phi_1(v) , \qquad y = \psi(u) + \psi_1(v) , \qquad z = \chi(u) + \chi_1(v) .$$

The surface thus defined is *a surface of translation* (cf. no. 227). It is of two types: the displacement of the curve $x = \phi(u)$, $y = \psi(u)$, $z = \chi(u)$, or the displacement of $x = \phi_1(v)$, $y = \psi_1(v)$, $z = \chi_1(v)$.

II. NON-DEVELOPABLE RULED SURFACES

246. Various forms of the equations. The spherical indicatrix and spherical curves.

A ruled surface is the locus of a line depending on a parameter. We assume that this line does not have an envelope, hence the surface is non-developable. However, the following will apply to a developable surface which is not a cylinder.

In Cartesian coordinates, a ruled surface S will be given by

$$x = az + p , \qquad y = bz + q , \tag{6}$$

where a,b,p,q are functions of a parameter u, when the generators (understood to be rectilinear) are not parallel to Oxy; otherwise, we could take $z = q$, $y = mx + p$, where q,m and p are functions of one parameter. We have seen (no. 61) that the surface defined by the equations (6) is only developable if $a'q' - b'q' \equiv 0$.

In the parametric form, the generators correspond to $v = \text{const.}$, we can take

$$x = \xi + \alpha u , \qquad y = \eta + \beta u , \qquad z = \zeta + \gamma u , \tag{7}$$

where $\xi,\eta,\zeta,\alpha,\beta$ and γ are functions of v alone. α,β,γ are the direction parameters of the generator; ξ,η,ζ are the coordinates of a point M of a

curve traced on the surface, which will be known as the *directrix curve*. The equations (6) enter into the general case (7). The vectorial expression condensing the formulae in (7) is

$$\vec{P} = \vec{M} + u\vec{\ell} \ , \tag{8}$$

where $\vec{\ell}$ is the vector with components α,β,γ.

THE DIRECTION CONE AND THE SPHERICAL INDICATRIX

If through a fixed point, that could be taken to be the origin, we take parallels to the generators of the surface S, then these parallels will engender a cone (since S is not a cylinder). This is the direction cone of the surface, having 0 as the vertex. The assignment of this cone and a direction curve as a function of v, defines S.

Let us take through 0, a unit vector equipollent to the unit vector of the generator with the direction of $\vec{\ell}$. The extremity m of this vector describes a spherical curve Λ on the unit sphere Σ. *This curve* Λ *is the spherical indicatrix of the surface* S. It changes to a symmetric curve when the positive direction is changed on the generators. The direction cone is the cone with vertex 0 and directrix Λ.

In order to study S, it will be convenient to assume that the vector $\vec{\ell}$ is a unit vector; then $\overrightarrow{Om} = \vec{\ell}$ and the parameter v is the arc of Λ. Under the form (7), we will have $\alpha^2 + \beta^2 + \gamma^2 = 1$, $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$. The indicatrix will be defined by

$$\vec{Om} = \vec{\ell} \ , \qquad |\vec{\ell}| = 1 \ , \qquad \left|\frac{d\vec{\ell}}{dv}\right| = 1 \ . \tag{9}$$

THE FRENET-SERRET FORMULAE FOR A SPHERICAL CURVE

Let Λ be a spherical curve defined by the conditions in (9). The trihedron of Darboux-Ribaucour at the point m is defined by the vector $\vec{t}$ of the tangent, the vector $\vec{\ell}$ normal to the sphere and $\vec{g}$ such that $\vec{\ell} = \vec{t} \wedge \vec{g}$ (no. 231). Since we already have

$$\vec{t} = \frac{d\vec{\ell}}{dv} \quad ,$$

the third equation (46) of no. 231 yields

$$\vec{t} = -\frac{\vec{t}}{R_N} - \frac{\vec{g}}{T_g} \ , \qquad \frac{1}{T_g} = 0 \ , \qquad R_N = -1 \quad .$$

The fact that the geodesic torsion is zero, relates to the fact that every spherical curve is a line of curvature of the sphere, $R_N = -1$ translates, for the sphere of radius 1, into Meusnier's theorem. The formulae (46) of no. 231 are then written, on setting $R_v = \rho$, as

$$\frac{d\vec{t}}{dv} = \frac{\vec{g}}{\rho} - \vec{\ell}, \qquad \frac{d\vec{g}}{dv} = -\frac{\vec{t}}{\rho}, \qquad \frac{d\vec{\ell}}{dv} = \vec{t}, \tag{10}$$

where the first two equations are the Frenet-Serret formulae for a spherical curve.

247. The line element. The variation of the tangent plane. The central point. The line of striction.

Formula (8) yields

$$d\vec{P} = d\vec{M} + \vec{\ell}du + u\vec{t}dv \quad .$$

Let us decompose the vector $\overrightarrow{dM}$ on the axes of the Darboux-Ribaucour tihedron at the point m, by writing

$$\frac{d\vec{M}}{dv} = -c\vec{t} + k\vec{g} + \lambda\vec{\ell} \quad , \tag{11}$$

where c, k and λ are functions of v. The condition $\lambda \equiv 0$ would imply that the directrix curve is constantly orthogonal to the generators. We then have

$$\overrightarrow{dP} = \overrightarrow{P'_u}du + \overrightarrow{P'_v}dv$$

$$\vec{P}'_u = \vec{\ell}, \qquad \vec{P}'_v = (u-c)\vec{t} + k\vec{g} + \lambda\vec{\ell} \;. \tag{12}$$

The line element is given by

$$ds^2 = \overrightarrow{dP}^2 = [(u-c)^2+k^2]dv^2 + (du+\lambda dv)^2 \quad .$$

When $\lambda \equiv 0$, the curves $u = \text{const.}$ are all orthogonal to the generators and the ds^2 takes the geodesic form.

The variation of the tangent plane. The theorem of Chasles.

The tangent plane at the point P is defined by the vectors (12); its trace onto the tangent plane at m to the sphere Σ is defined by

$$(u-c)\vec{t} + k\vec{g} \quad . \tag{13}$$

If $k = 0$, this plane is the same along the generators; this is the plane $\vec{t}, \vec{\ell}$. *Hence, if* $k \equiv 0$, *the tangent plane is the same along each generator; it only depends on* v *and the surface is developable.* This sufficient condition is also necessary. When $k \neq 0$, the tangent plane turns about the generator and the surface is not developable.

If k is not identically zero, we could have $k = 0$ for certain values of v. The tangent plane is the same along a generator corresponding to such a value. We refer to it as a *stationary generator.*

Let us assume $k \neq 0$. For $u = c$, the tangent plane is the plane $\vec{g}$, $\vec{\ell}$. For $u \neq c$, the tangent plane makes an angle ω with the plane $\vec{g}$, $\vec{\ell}$ and we have, following (13), by taking this angle positively from $\vec{g}$ towards $\vec{t}$,

$$k = \mu \cos \omega \,, \qquad u - c = \mu \sin \omega \,,$$

hence

$$\mathrm{tg}\omega = \frac{u-c}{k} \quad . \tag{14}$$

THIS IS THE FORMULA OF CHASLES which shows that, when u varies from $-\infty$ to $+\infty$, the tangent plane turns through π about the generator. At the point at infinity, the tangent plane is the plane $\vec{t}$, $\vec{\ell}$; it is perpendicular to the tangent plane at the point P_c of the generator which corresponds to $u = c$. *This point* P_c *is the* CENTRAL POINT. The relation between a point P of the generator and the tangent plane at this point is a homographic relationship ideally defined by the equality in (14). The number $1/k$ is called the PARAMETER OF THE DISTRIBUTION (understood to be the tangent planes), the tangent plane at the central point is THE CENTRAL PLANE; the tangent plane at the point at infinity is the *asymptotic plane.*

When $k > 0$, the tangent plane turns from the left to the right of an observer situated on the generator and moved from head to toe through the vector $\vec{\ell}$ when the point of contact is taken in that opposite direction. This does not change when we replace the vector $\vec{\ell}$ by the opposite vector (since $\vec{g}$ is then invariant and $\vec{t}$ is replaced by $-\vec{t}$ in (11) and k does not change). Similarly, changing v to $-v$ does not change k.

LINE OF STRICTION

The line described by the central points is known as the line of striction.

THE CALCULATION OF c AND k

The formula (11) multiplied scalar-wise by $\vec{t}$ and $\vec{g}$ yields

$$c = -\frac{d\vec{M}}{dv}\vec{t} \,, \qquad k = \frac{d\vec{M}}{dv}\vec{g} \quad .$$

Now $\vec{g} = \vec{\ell} \wedge \vec{t}$, and $\vec{t}$ is given by the third formula in (10), hence

$$c = -\frac{d\vec{M} d\vec{\ell}}{dv^2} \,, \qquad k = \frac{(\vec{\ell}, d\vec{\ell}, d\vec{M})}{dv^2} \quad .$$

If the surface is given by the formulae in (7), where v is not necessarily the arc of the curve, but $\vec{\ell}$ is a unit vector, $\alpha^2 + \beta^2 + \gamma^2 = 1$, then we will have

$$c = -\frac{\Sigma\xi'\alpha'}{\Sigma\alpha'^2}\ , \qquad k = \frac{\|\alpha,\alpha',\Sigma'\|}{\Sigma\alpha'^2}\ . \tag{15}$$

THE CALCULATION OF $1/\rho$

The geodesic curvature of the spherical indicatrix is given by the first formula (10). We have

$$\frac{1}{\rho} = \vec{g}\,\frac{d\vec{t}}{dv} = (\vec{\ell}\wedge\vec{t})\,\frac{d\vec{t}}{dv} = \frac{(\vec{\ell},d\vec{\ell},d^2\vec{\ell})}{dv^3} = \frac{\|\alpha,\alpha',\alpha''\|}{(\Sigma\alpha'^2)^{3/2}}\ . \tag{16}$$

Remarks. I. If the surface is developable (without being a cylinder), $k\equiv 0$, then the point P_c is the point of the generator where the tangent plane is indeterminate; this is the characteristic point of the generator.

II. If the surface is taken to be of the form in (6), then the formulae (15) yield

$$k = \frac{(a'q'-b'p')(a^2+b^2+1)}{(a'^2+b'^2)+(ab'-ba')^2}\ ,$$

$$z_c = -\frac{a'p'+b'q'+(ab'-ba')(aq'-bp')}{(a'^2+b'^2)+(ab'-ba')^2}\ ,$$

where z_c denotes the central point.

Theorem. *The central point of a generator* Δ *is the limiting position of the foot of the perpendicular on* Δ*, common to* Δ *and to a neighboring generator* Δ'*, when* Δ' *tends towards* Δ.

Let P and P' be the feet of this common perpendicular on Δ and Δ'. We have

$$\vec{P} = \vec{M} + u\vec{\ell}\ , \qquad \vec{P}' = \vec{M} + d\vec{M} + u(\vec{\ell}+d\vec{\ell}) + \vec{\ell}\delta u\ ,$$

where u and v are the coordinates of P and $u+\delta u$, $v+\delta v$ those of P'. Hence

$$\vec{P}' - \vec{P} = \overrightarrow{PP'} = \overrightarrow{dM} + ud\vec{\ell} + \vec{\ell}\delta u = [(u-c)\vec{t} + k\vec{g} + \lambda\vec{\ell}]dv + \vec{\ell}\delta u\ , \tag{17}$$

and we obtain the conditions

$$\overrightarrow{PP'}\vec{\ell} = 0, \qquad \overrightarrow{PP'}(\vec{\ell}+d\vec{\ell}) = \overrightarrow{PP'}(\vec{\ell}+\vec{t}dv) = 0\ ;$$

the second condition yields, on account of the first,

$$\overrightarrow{PP'}\vec{t} = 0\ .$$

On replacing $\overrightarrow{PP'}$ by its value, we see that we must have

$$\lambda dv + \delta u = 0 \,, \qquad (u-c)dv = 0 \,, \tag{18}$$

which implies $u = c$ and so proves the theorem.

THE PARTICULAR CASE OF SURFACES WITH A DIRECTION PLANE

In this case, where the generators are parallel to a fixed plane which may be taken to be the plane Oxy (where Oxyz is tri-rectangular), *the orthogonal projection of the line of striction onto the plane* Oxy *is the envelope of the projection of the generators onto this plane.* Since, given two neighboring generators with δ and δ' their projections onto Oxy, then the common perpendicular to Δ and Δ' is parallel to Oz passing through the common point of δ and δ'. When Δ' tends toward Δ, δ' tends towards δ, and the common point of δ and δ' has as its limit, the point of contact of δ with its envelope.

We can also say that the line of striction is the locus of points of the surface where the tangent plane is perpendicular to the plane Oxy, which also results from the fact that the asymptotic plane is parallel to Oxy.

Remark. On taking account of the equations in (18), we see that (17) can be written as

$$\overrightarrow{PP'} = k\vec{g}dv \quad ,$$

and we have therefore

$$|k| = \lim_{dv\to 0} \left|\frac{\overrightarrow{PP'}}{dv}\right| \quad .$$

248. Asymptotic lines

One of the system of asymptotic lines of a ruled surface is formed by generators. The equation of the asymptotic lines must contain dv as a factor, and the determinant D must be zero. On account of the formulae in (12), we have, in fact,

$$D = [\vec{\ell},(u-c)\vec{t} + k\vec{g} + \lambda\vec{\ell},0] = 0 \quad ,$$

since $\vec{\ell}$ does not depend on u. Consequently, we have

$$D' = \left(\vec{\ell},(u-c)\vec{t} + k\vec{g}, \frac{d\vec{\ell}}{dv}\right) \quad ,$$

$$D'' = (\vec{\ell},(u-c)\vec{t} + k\vec{g}, \vec{P}''_{vv}) \,,$$

where the term $\lambda\vec{\ell}$ has been suppressed in these mixed products. On account of the formula in (10), we have

$$D' = (\vec{\ell},(u-c)\vec{t} + k\vec{g},\vec{t}) = -\vec{g}((u-c)\vec{t} + k\vec{g}) = -k \quad ,$$

$$\vec{P}''_{vv} = -c'\vec{t} + k'\vec{g} + \lambda'\vec{\ell} + (u-c)\left(\frac{\vec{g}}{\rho} - \ell\right) + \vec{t}\left(\lambda - \frac{\vec{k}}{\rho}\right) \quad ;$$

$$D'' = ((u-c)\vec{g} - k\vec{t})\vec{P}''_{vv} = (u-c)\left[k' + (u-c)\frac{1}{\rho}\right] + k\left(c' + \frac{k}{\rho} - \lambda\right) .$$

The differential equation of the second system of asymptotic lines is

$$2k\,\frac{du}{dv} = \frac{1}{\rho}(u-c)^2 + k'(u-c) + k\left(c' + \frac{k}{\rho} - \lambda\right) \quad ,$$

which is a Riccati equation in the general case where $\frac{1}{\rho}$ is nonzero. We can only integrate it, in general, when we know a solution. Following the property of the solutions of the Riccati equations (no. 67), we see that if we take four solutions u_1, u_2, u_3, u_4 corresponding to four values C_1, C_2, C_3, C_4 of the constant of integration, then the bi-ratio (u_1,u_2,u_3,u_4) will be constant and equal to the bi-ratio of these values of the constant. The corresponding lines L_1, L_2, L_3, L_4 will intersect a generator Δ in four points P_1, P_2, P_3, P_4 whose bi-ratio is that of the values u_1, u_2, u_3, u_4. It is, therefore, the same, whatever the generator Δ.

THE CASE WHERE THE SURFACE IS GIVEN BY THE EQUATIONS (6)

In this case, the axes can be taken to be arbitrary, and the equation of the asymptotic lines, other than the generators, is provided at once by

$$\begin{vmatrix} a & b & 1 \\ a'z + p' & b'z + q' & 0 \\ (a''z+p'')+2a'\,\frac{dz}{du} & (b''z+q'')+2b'\,\frac{dz}{du} & 0 \end{vmatrix} = 0 \quad ,$$

where

$$2(a'q'-b'p')\,\frac{dz}{du} + (a''z+p'')(b'z+q') - (b''z+q'')(a'z+p') = 0 \quad . \qquad (20)$$

PARTICULAR CASES

I. If $\frac{1}{\rho}$ is zero, then equation (19) becomes a linear equation and is integrated by quadratures. This corresponds to the case where the spherical indicatrix is a geodesic of the sphere, hence a great circle, and the surface admits a direction plane. We arrive at the same conclusion on commencing from equation (20) which becomes linear when $a''b' - a'b'' = 0$, which yields $a' = hb'$, where h is a constant, and then $a = hb + h'$, where h' is another constant. This indeed expresses the fact that the lines in (6) are parallel to a plane.

II. Under the guise of (20), we see that the equation reduces to a Bernouilli equation of $p''q' - p'q'' = 0$, hence $p' = Cq'$, $p = Cq + C'$, where C and C' are constants. The section of the surface through the plane $z = 0$, is a line.

This line is an asymptotic line, and equation (20) happens to be satisfied for $z=0$ and reduces to quadratures.

The analogous situation to equation (19) arises when

$$\frac{c^2}{\rho} - k'c + k\left(c' + \frac{k}{\rho} - \lambda\right) \equiv 0 \quad . \tag{19}$$

This condition does not imply that the directrix curve $u=0$ is a line, but only the fact that, at each point M of the directrix curve Γ, the vector $\vec{\ell}$ is situated in the osculating plane of Γ.

III. When the above two situations arise simultaneously, then matters are reduced to a linear homogeneous equation, which is, in fact,

$$2k \frac{du}{dv} = k'u \quad ,$$

and can be integrated at once. We have

$$u^2 = Ck \quad .$$

(cf. the result obtained for the conoids in no. 241).

III. CURVES AND SURFACES ASSOCIATED TO CONGRUENCES OF CURVES AND LINES

249. Curve congruences. The focal surface.

By a congruence of curves in three-dimensional space, we mean a family of curves depending on two parameters λ,μ, such that, in terms of a three-dimensional domain of the space, there exists at least one curve of the family passing through an arbitrary point. We can also say that it is to be assumed that these curves are not situated on one surface.

These curves will be defined by $\vec{M}(u,\lambda,\mu)$ and we assume that this vector possesses the necessary derivatives for the calculations in question. For λ and μ fixed, the point M will describe a curve $\Gamma(\lambda,\mu)$ of the congruence. In cartesian coordinates, we will have

$$x = f(u,\lambda,\mu) \ , \qquad y = g(u,\lambda,\mu) \ , \qquad z = h(u,\lambda,\mu) \ . \tag{21}$$

and we assume that we do not have

$$\frac{D(x,y,z)}{D(u,\lambda,\mu)} \equiv 0 \quad . \tag{22}$$

To a system of values u, λ, μ, there corresponds a point $M(x,y,z)$, which is unique, when we consider a domain of the u, λ, μ space where f, g, h are uniform. If, for this system of values, the functional determinant of the

first member of (22) is nonzero, then one and only one curve $\Gamma(\lambda,\mu)$ will pass through the neighboring points of M. This curve corresponds to the values of λ, μ and u, near to those defining M. Within a small domain, we have a regular congruence of curves without common points which cannot be tangential to a curve or to a surface. We can associate these curves to form surfaces by assuming that λ and μ are subjected to some relation, $k(\lambda,\mu) = 0$, where the function k admits partial derivatives.

The *singular points* of the congruence are the points M given by the values λ, μ and u for which the above discussion no longer applies. In particular, they are points such that

$$\frac{D(x,y,z)}{D(u,\lambda,\mu)} = 0 \quad . \tag{23}$$

These singular points could constitute lines or surfaces, or even be isolated. They might also correspond to either a finite or infinite number of values of λ and μ.

THE CONGRUENCES DEFINED BY THE INTERSECTIONS OF SURFACES

We shall regard the congruence as defined by the intersection of two surfaces depending on two parameters, as a preferential viewpoint in our theoretical study:

$$\Phi(x,y,z,\lambda,\mu) = 0 \ , \qquad \Psi(x,y,z,\lambda,\mu) = 0 \quad ; \tag{24}$$

the functional determinant

$$\frac{D(\Phi,\Psi)}{D(\lambda,\mu)} \tag{25}$$

is not identically zero for the systems of values satisfying the equations (24).

If x,y,z,λ,μ is a system of numbers which satisfy the equations in (24) and for which the determinant (25) is nonzero, then we may solve the system in (24) in λ and μ in the neighborhood of these values and restate it in the equivalent form

$$K(x,y,z) = \lambda \ , \qquad L(x,y,z) = \mu \quad . \tag{26}$$

We obtain a regular congruence in the neighborhood of the point $M(x,y,z)$ when it is assumed that the surfaces (24) effectively intersect along curves. This happens to be the case when the functional determinants of Φ and Ψ with respect to x,y or to y,z or to z,x are not all three simultaneously zero at the point M.

The singular points correspond to the case where the equations (24) are satisfied, where the functional determinant (25) is zero. There cannot

exist any singular point at a finite distance; this is the case for the congruence of lines parallel to one line. There cannot exist any singular point at which the curves of the congruence are ordinary; this is the case for the congruence defined by

$$z = \lambda \ , \qquad \log(x^2+y^2) - \operatorname{arc\,tg} \frac{y}{x} - \mu = 0$$

which is formed by logarithmic spirals.

THE FOCAL SURFACE IN THE GENERAL CASE

Let us consider the following case, which is the general case when Φ and Ψ are polynomials with respect to the five variables occurring, and the elements are taken to be either real or complex. *On eliminating* λ *and* μ *between the relations (24) and*

$$\frac{D(\Phi,\Psi)}{D(\lambda,\mu)} = 0 \quad , \tag{27}$$

we obtain a single relation between x, y, z. This amounts to saying that in terms of λ and μ, we can solve the system formed by the equation (27) and by one of the equations (24). On taking the values so obtained into the other equation (24) we obtain the relation in question

$$R(x,y,z) = 0 \quad , \tag{28}$$

which represents a surface Σ. *More precisely*, taking Δ to denote the functional determinant of the first member of (27), we assume that, e.g., we have

$$\frac{D(\Delta,\Phi)}{D(\lambda,\mu)} \neq 0 \quad .$$

This implies that one of the partial derivatives of Φ, e.g. that relative to λ, is nonzero. By considering a small domain, we could write the first equation (24) on solving with respect to λ, and replace λ by this value in the second equation. *The system (24), (27) can be written in the form*

$$\lambda = \Phi(x,y,z,\mu) \ , \qquad \Psi(x,y,z,\mu) = 0 \ , \qquad \frac{\partial\Psi}{\partial\mu} = 0 \quad , \tag{29}$$

along with the hypothesis that the functional determinant of the first and last equation is nonzero, hence that

$$\frac{\partial^2\Psi}{\partial\mu^2} \neq 0 \quad .$$

The equation (28) of the surface Σ is obtained by eliminating μ between the last two equations (29). If we assume that one of the functional determinants of these last two equations with respect to x,y, or y,z, or z,x

is nonzero, then the surface Σ is an envelope of the surfaces $\Psi = 0$ (no. 59). The curves Γ of the congruence, which are on $\Psi = 0$, are tangent to Σ. Thus, on returning to the system (24), (27):

If the equations (24) and (27) permit the regular elimination of λ *for example, with* $\frac{\partial\Phi}{\partial\lambda}$ *nonzero, and if the two equations*

$$\Psi = 0 , \qquad \frac{D(\Phi,\Psi)}{D(\lambda,\mu)} = 0$$

are solvable with respect to two of the variables x,y,z, *with nonzero functional determinant, then the surface* Σ *defined by the system is an envelope of the curves of the congruence.* We say that Σ is a *focal surface.* Its points are called *foci* or *focal points.*

Let us take the system to be of the form (29).

Through each point M of Σ there passes a curve Γ whose tangent is the intersection of the tangent plane to Σ (which coincides with the tangent plane to $\Psi = 0$) and of the tangent plane to the surface $\Phi = \lambda$. We thus define at each point M of Σ a direction $T(M)$, namely, a continuous differentiable function of the position of M. There exists on Σ a family of parameter curves, tangent at each point of M to the direction $T(M)$, and hence to the curve of the congruence passing through M. Thus

The focal surface Σ is also the locus of the enveloping curves of the curves of the congruence when suitably chosen.

A DIRECT STUDY OF THE ENVELOPING CURVES OF THE CONGRUENCE

If we take the curves to be of the general form (24), then we will obtain a family of curves of the congruence depending on a single parameter by assuming that μ is a function of λ. These 1-parameter curves will have an envelope when the system of equations obtained by adding to the equations in (24) their derived equations with respect to λ, i.e.

$$\frac{\partial\Phi}{\partial\lambda} + \frac{\partial\Phi}{\partial\mu}\frac{d\mu}{d\lambda} = 0 , \qquad \frac{\partial\Psi}{\partial\lambda} + \frac{\partial\Psi}{\partial\mu}\frac{d\mu}{d\lambda} = 0 , \tag{30}$$

is compatible for any λ (no. 61). It is therefore necessary for x,y,z,λ to satisfy, at once, the equations (24) and the equation (27) obtained by eliminating $\frac{d\mu}{d\lambda}$ from the equations (30). *The points of contact of the curves* Γ *with their envelopes can only be singular points.*

The system of equations (24) and (30) is equivalent to the system obtained by replacing one of the equations (30) by the equation (27); when $\frac{\partial\Phi}{\partial\mu}$ is nonzero, we will consider the system

$$\Phi = 0 , \qquad \Psi = 0 , \qquad \frac{\partial\Phi}{\partial\lambda} + \frac{\partial\Phi}{\partial\mu}\frac{d\mu}{d\lambda} = 0 , \qquad \frac{D(\Phi,\Psi)}{D(\lambda,\mu)} = 0 . \tag{31}$$

(If $\frac{\partial\Phi}{\partial\mu}$ and $\frac{\partial\Psi}{\partial\mu}$ happened to be identically zero, then Φ and Ψ would only depend on λ, and we would not get a congruence). Elimination of x,y,z from the equations (31) then leads to the differential equation

$$\Theta\left(\lambda,\mu,\frac{d\mu}{d\lambda}\right) = 0 \tag{32}$$

defining the functions μ of λ for which the curves of the congruence admit an envelope. In the case where Φ and Ψ are polynomials, this elimination can be regarded as algebraic and equation (32) will be algebraic. In the general case, *where the existence of a focal surface is assumed*, we can take the equations of the congruence and the focal surface in the reduced form (29); it is sufficient to include the condition

$$1 = \frac{\partial\Phi}{\partial\mu}\frac{d\mu}{d\lambda} \quad , \tag{33}$$

along with these equations.

Remarks. I. If we consider the direction $T(M)$ as defined above, then we see that it is given by

$$\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}\ dz = 0 \quad , \tag{34}$$

where x,y,z is a point of Σ and λ and μ satisfy the equations in (29). The relation between λ and μ defining $T(M)$ is obtained by differentiating the first equation (29). On taking account of (34), we recover the relation (33).

II. Every curve Γ of the congruence containing a focal point can, in general, be associated to other curves of the congruence to constitute a family admitting an envelope. This is a consequence of the above discussion concerning the existence of a focal surface. We shall consider the degenerate case in no. 250.

FOCAL POINTS SITUATED ON A CURVE OF THE CONGRUENCE

A curve Γ could contain several focal points which are ordinary points of Γ. Let such points be denoted by $M_1,M_2,\ldots,M_p$. At each M_j, the curve Γ will be, in the general case, tangent to a *focal surface* (or to a sheet of a focal surface) Σ_j, and to a curve envelope C_j. The tangent curves to C_j, which comprise Γ, form a surface S_j, which in general, admits C_j as a singular line. The surfaces S_j, are distinct, in general: the curves of the congruence tangent to C_j are not tangent to C_k, $k \neq j$. It follows that the surface S_j will be tangent to Σ_k along a curve intersecting C_k and whose points will be ordinary points of S_j (fig. 82). At the point of contact M_k of Γ and Σ_k, the tangent plane of S_j is identified with the

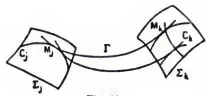

Fig. 82.

tangent plane to Σ_k, whilst at the point of contact M_j of Γ with Σ_j, the tangent plane to S_j and the tangent plane to Σ_j will not be identified, in general.

A REMARK ON THE ANALYTIC CASE

In this case, the elimination to equation (28) will, in general, lead to a single analytic function $R(x,y,z)$. The various portions of the focal surfaces will be sheets of a single surface. This, in particular, arises in the algebraic case. In the nonanalytic case, there does not exist any relation between the portions of the focal surfaces without common points.

250. The degenerate case. The case of parametric equations.

Let us consider the case whereby elimination of λ and μ from equations (24) and (27) yield two relations between x,y,z. We may again assume that, e.g., $\frac{\partial\Phi}{\partial\lambda}$ is nonzero on the curve thus obtained, since Φ and Ψ cannot be independent of λ and μ on this curve. We could then consider the equations in the form (29), but these equations must be satisfied on a curve. Here, we have

$$\frac{\partial^2\Psi}{d\mu^2} \equiv 0 \quad ,$$

for the focal points in question, and the equation

$$\frac{\partial\Psi}{\partial\mu} = 0$$

has a first member independent of μ. Moreover, on account of this equation, the first member of the equation $\Psi = 0$ must also be independent of μ. We will obtain a line of focal points and for a point x,y,z on this line, the equations of the curves Γ at this point, reduce to

$$\lambda = \Phi(x,y,z,\mu) \ , \qquad \Psi_1(x,y,z) = 0 \ .$$

There exists an infinity of curves Γ, *depending on one parameter, passing through each point of this focal curve.*

Example. Consider the congruence

$$y - \mu x = 0 , \qquad z - \lambda + \Phi(x,y,z,\mu) = 0 ,$$

with the condition $\Phi(0,0,\lambda,\mu) \equiv 0$. Equation (27) reduces to $x = 0$, which gives $y = 0$. For the curves intersecting the axis Oz, μ is arbitrary and $\lambda = z_0$, where z_0 denotes the point of intersection.

Remark. In the degenerate case, finding curves Γ admitting an envelope, via the equations (29), will lend to equation (33). On letting $\Phi_1(z,\mu)$ denote the function $\Phi(x,y,z,\mu)$ when x and y are replaced by their expressions as functions of z along the focal curve, then we have

$$1 - \frac{\partial \Phi_1}{\partial \mu} \cdot \frac{d\mu}{d\lambda} = 0 \qquad \text{with} \qquad \lambda = \Phi_1(z,\mu) \quad .$$

The differential equation obtained by eliminating z from these two equations, is the differential equation of the curves $\lambda = \Phi_1(z,\mu)$. Thus the only way of associating curves Γ which admit an envelcpe belonging to the focal curve is to consider those curves Γ passing through the various points of this curve, where the relation between λ and μ is $\lambda = \Phi_1(z,\mu)$.

THE CASE OF ISOLATED FOCAL POINTS

In this case, all of the curves Γ pass through a fixed point. Every one parameter family formed by the curves Γ, can be considered as admitting this fixed point as an envelope.

Examples. I. The circles Γ tangent to a sphere Σ and which are orthogonal to a line D, form a congruence which admits Σ as a focal surface and D as a focal line. On Σ, the envelopes of the circles Γ are the sections of Σ by the planes passing through D.

II. The lines Γ which intersect a line D and which are tangent to a sphere Σ form a congruence for which D is a focal line and Σ a focal surface. The envelopes of the lines Γ on Σ, are circular sections of Σ through the planes passing through D; the lines Γ which pass through a point of D constitute a cone of revolution. A polar inversion A outside of D and Σ transforms this congruence intc a congruence of circles admitting a focal surface, a focal curve and a focal point, namely the point A.

THE CASE OF PARAMETRIC EQUATIONS

When the congruence is taken to be of the form (21), the singular points are obtained by eliminating λ,μ and u from equations (21) and (23). We may thus obtain the sheets of the focal surface, the curves and the points. In order to obtain the envelopes of the curves $\Gamma(\lambda,\mu)$ of the congruence, we must consider μ as a function of λ in such a way that the curve defined by

$$\vec{M}(u,\lambda,\mu(\lambda))$$

has an envelope. We will have (no. 61)

$$\vec{M}'_u \wedge \left(\vec{M}'_\lambda + \vec{M}'_\mu \frac{d\mu}{d\lambda}\right) = 0 \quad . \tag{35}$$

On scalar multiplication by $\vec{M}'_\mu$ we recover condition (23) under the form

$$(\vec{M}'_u, \vec{M}'_\lambda, \vec{M}'_\mu) = 0 \quad .$$

On account of this condition, equation (35) will reduce to one of the two equations

$$\frac{\dfrac{\partial f}{\partial \lambda} + \dfrac{\partial f}{\partial \mu}\dfrac{d\mu}{d\lambda}}{\dfrac{\partial f}{\partial u}} = \frac{\dfrac{\partial g}{\partial \lambda} + \dfrac{\partial g}{\partial \mu}\dfrac{d\mu}{d\lambda}}{\dfrac{\partial g}{\partial u}} = \frac{\dfrac{\partial h}{\partial \lambda} + \dfrac{\partial h}{\partial \mu}\dfrac{d\mu}{d\lambda}}{\dfrac{\partial h}{\partial u}} \quad .$$

251. The comparison with the theory of singular integrals of differential systems.

When we considered a system of two first order differential equations of the form

$$P(x,y,z,y',z') = 0 \; , \qquad Q(x,y,z,y',z') = 0 \; , \tag{36}$$

as studied in no. 149, we saw that the singular integrals, if they exist, are the envelopes of general integral curves which depend on two parameters and that these singular integral curves are situated on surfaces. We see that these surfaces are focal surfaces of the congruence formed by the integral curves. In no. 149, we confined ourselves to the most elementary examples; this is why we had not questioned the case whereby the congruence admits a focal curve. In this case of a focal curve, the system of equations is indeterminate at each point of this curve when y' and z' are regarded as unknowns. On the other hand, in no. 149, we discovered that the surface on which the Cauchy theorem does not apply, is in general a locus of turning points of integral curves; in contrast, in the above discussion, we confined ourselves to the case where the focal points are ordinary points of the curves of the congruence.

The two points of view are complete. The situation encountered in no. 149 can arise for a congruence given *a priori*; the systems of differential equations may admit higher order singularities for which there is indetermination.

252. Line congruences

In the case of line congruences, we may, having chosen the axes suitably, assume that these lines are not parallel to the plane Oxy in general. The equations can be written in the form

$$x = az + p\ , \qquad y = bz + q\ ,$$

where a,b,p,q are functions of two parameters which we will hereby denote as u and v. The singular points will satisfy these equations and the condition

$$\begin{vmatrix} a'_u z + p'_u & a'_v z + p'_v \\ b'_u z + q'_u & b'_v z + q'_v \end{vmatrix} = 0\ .$$

The relation between u and v for which the corresponding lines have an envelope is given by

$$\begin{aligned} a'_u z + p'_u + (a'_v z + p'_v)\frac{dv}{du} &= 0\ , \\ b'_u z + q'_u + (b'_v z + q'_v)\frac{dv}{du} &= 0\ . \end{aligned} \tag{39}$$

When we eliminate $\frac{dv}{du}$ from these equations, we obtain condition (38). When this condition is satisfied, $\frac{dv}{du}$ is given by the relation obtained on eliminating z from the equations (39), which yields the differential equation

$$\begin{vmatrix} a'_u du + a'_v dv & b'_u du + b'_v dv \\ p'_u du + p'_v dv & q'_u du + q'_v dv \end{vmatrix} = 0\ , \tag{40}$$

or, in short,

$$dadp - dbdp = 0\ . \tag{41}$$

On account of (38), a line of the congruence (corresponding to a pair u,v) contains in general two singular points, which are in general foci. Their denotations z are the roots of the second degree equation (38); the values of x and y are given by the equalities (37). For certain values of u and v, equation (38) in z could have an infinite root, roots identified, or could be satisfied for any z.

Example. Let us consider the lines which intersect the two circles (rectangular axes assumed).

$$z = h\ , \quad x^2 + y^2 - a^2 = 0\ ; \qquad z = -h, \quad x^2 + y^2 - b^2 = 0\ ,$$

where $a \neq b$ and h are nonzero constants. These lines have as their equations

$$2h(x - a\cos u) + (z - h)(b\cos v - a\cos u) = 0 \quad ,$$

$$2h(y - a\sin u) + (z - h)(b\sin v - a\sin u) = 0 \quad .$$

Equation (38) is here

$$(z - h)(z + h)\sin(u - v) = 0 \quad . \tag{42}$$

For $z = h$ and $z = -h$, we obtain the points of the two circles; we have arrived at the degenerate case (no. 250). For $u = v$, the equations of the convergence are satisfied and yield the lines

$$x\sin u - y\cos u = 0, \qquad 2hx - \cos u(a(z+h) - b(z-h)) = 0 \quad .$$

These are the generators of one of the cones of revolution which pass through the two given circles. The other cone corresponds to $u = v + \pi$. All of the points of these cones are singular (but are not focal points): the regular solution of the system of equation (42) and one of the equations of the congruence is impossible.

THE DEVELOPABLE SURFACES OF THE CONGRUENCE

When condition (38) is realized, we take u to be a solution of equation (41). We obtain a one parameter family of lines of the congruence which admit an envelope. These lines engender a developable surface. We say that it is *a developable surface of the congruence*. On account of the general result of nos. 249 and 250, the edge of regression of this surface, on the vertex of the cone if it reduces to a cone, belong to the locus of the *singular* points. In general they will belong to a focal surface or to a focal curve in the case of a cone, but it may also be the case that it does not belong to a focal surface nor to a focal curve nor identified with a focal point. Following the terminology of Julia (see his Course de géométrie) we say that we have *singular developable* of the congruence. The lines belonging to this developable are *singular lines*, i.e. *the loci of singular points*. This was the case in the above example for cones passing through the two circles. If, in this example, we replace one of the circles by an ellipse, we will obtain two singular developables, not reducible to cones. We have analogous results for all congruences formed by the common tangents to two surfaces or by lines intersecting two curves.

THE DEVELOPLABLES OF THE CONGRUENCE PASSING THROUGH AN ORDINARY LINE

An ordinary line of the congruence is a line containing two distinct foci. Elimination of u and v from equations (37) and (38) can be made by the usual means and shows (no. 249) that the line D is tangent to two curves situated on two sheets of the focal surface, in the general case where there

is no degeneracy. There exists two developable surfaces of the congruence which contain D. One has its edge of regression Λ_1 passing through the foci M_1; the other edge of regression Λ_2 is tangent to D at the second focus M_2. The first developable S_1 engendered by the tangents to Λ_1 which are tangents to the focal surface Σ_2 passing through M_2, is tangent to Σ_2, whereas the second developable surface S_2 is tangent at M_1 to the focal surface Σ_1 passing through M_1. The tangent planes to S_1 and S_2 along D are called *focal planes*, they are tangent planes to the focal surfaces at the points M_2 and M_1.

The *focal planes* can be obtained by determining the planes passing through D that D admit D as a characteristic. A plane passing through D has an equation of the form

$$X - aZ - p = \rho(Y - bZ - q) \ , \tag{43}$$

where ρ is a function of u and v is a function of u; we require the envelope of this plane. The characteristic is defined by equation (43) and by

$$-daZ - dp = d\rho(Y - bZ - q) - \rho(dbZ + dq) \ .$$

This equation must be satisfied for any Z when $Y - bZ - q = 0$; we therefore obtain

$$da - \rho db = 0 \ , \qquad dp - \rho dq = 0 \ .$$

The compatibility condition is equation (41) and we have

$$\rho = \frac{da}{db} = \frac{dp}{dq} = \frac{a'_u + a'_v v'_u}{b'_u + b'_v v'_u} = \frac{p'_u + p'_v v'_u}{q'_u + q'_v v'_u} \ .$$

We shall replace v'_u by the roots of equation (40) (divided by du^2) and on taking this into (43) we shall obtain the two focal planes. Also, in terms of the equations in (39), we can state the contribution of the focal point in the expression for ρ.

VECTORIAL NOTATION

If we take the congruence to be of the form

$$\vec{M} = \vec{P} + \vec{\ell}_u \ ,$$

where $\vec{P}$ (of fixed origin O) and $\vec{\ell}$ are functions of two parameters α and β, then the ruled surfaces belonging to the congruence will be obtained by taking β to be a function of α. Such a surface will be developable if we can choose u to be a function of α in such a way that $d\vec{M} \wedge \vec{\ell} = 0$, or

$$d\vec{P} \wedge \vec{\ell} + ud\vec{\ell} \wedge \vec{\ell} = 0 \ . \tag{44}$$

On scalar multiplication by $d\vec{\ell}$, we obtain the condition

$$(\vec{\ell}, d\vec{\ell}, d\vec{P}) = 0$$

or

$$\left(\vec{\ell}, \vec{\ell}'_\alpha + \vec{\ell}'_\beta \frac{d\beta}{d\alpha}, \vec{P}'_\alpha + \vec{P}'_\beta \frac{d\beta}{d\alpha}\right) = 0 \quad ,$$

a second degree differential equation in $\frac{d\beta}{d\alpha}$. The corresponding values of u are given by (44); they provide the foci situated on the line in question. This equation in u can be obtained directly; equation (44) is also written as

$$\left(\vec{P}'_\alpha + u\vec{\ell}'_\alpha + (\vec{P}'_\beta + u\vec{\ell}'_\beta) \frac{d\beta}{d\alpha}\right) \wedge \vec{\ell} = 0 \quad .$$

We eliminate the term in $\frac{d\beta}{d\alpha}$ by scalar multiplication by its coefficient, which yields

$$(\vec{P}'_\alpha + u\vec{\ell}'_\alpha, \vec{P}'_\beta + u\vec{\ell}'_\beta, \vec{\ell}) = 0 \quad .$$

253. The condition for a line congruence to be a normal congruence

We have seen in no. 229 that the congruence of normals of a surface S, has the following property: the normalities, i.e. the developable surfaces of the congruence, which pass through the same normal, intersect orthogonally along this line. Otherwise said, the focal planes at the two points M_1, M_2, where the normal is tangent to the two sheets of the surface of the centers (namely planes which are tangent planes to this surface of the centers at M_1 and M_2), are rectangular. This property still holds true when the surface of centers decomposes. We are going to that, conversely:

If along each line of a congruence, the focal planes are rectangular, then this congruence is the congruence of normals to a family of parallel surfaces.

This statement implies that the focal points are distinct, in general, and that the focal curves or surfaces exist. In order to establish the proposition, let us assume first of all, that one of the sheets of the focal surface does not reduce to a curve; let Σ denote this sheet of the focal surface. The lines D of the congruence are tangent to Σ, they admit as envelopes, a sheaf of curves C of Σ. We can define a point P on Σ by taking as coordinate curves on Σ, the curves C, β = const. and curves α = const. which will be, for example, their conjugate lines (these are then the contact curves of Σ and of the second system of developable surfaces of the congruence). The lines D of the congruence are defined by

$$\vec{M} = \vec{P} + \vec{\ell}u \quad ,$$

where $\vec{\ell}$ is the unit vector of the tangent to the curve C passing through P. $\vec{P}$ and $\vec{\ell}$ are functions of α and β. *Let us determine* u *as a function of* α *and* β *such that the surface described by the point* M *thus*

obtained, admits D *as a normal. The vectors* $\vec{M}'_\alpha$ *and* $\vec{M}'_\beta$ *must be orthogonal to* D. *Now*

$$\vec{M}'_\alpha = \vec{P}'_\alpha + \vec{\ell}'_\alpha u + \vec{\ell}u'_\alpha$$

$$\vec{M}'_\beta = \vec{P}'_\beta + \vec{\ell}'_\beta u + \vec{\ell}u'_\beta \tag{45}$$

The scalar product of these vectors and $\vec{\ell}$, must be zero. As $\vec{\ell}$ is a unit vector, $\vec{\ell}\vec{\ell}'_\alpha$ and $\vec{\ell}\vec{\ell}'_\beta$ are zero, and there remains the conditions

$$(\vec{P}'_\alpha\vec{\ell})'_\beta = (\vec{P}'_\beta\vec{\ell})'_\alpha \quad , \tag{46}$$

and, if this condition is realized, $u(\alpha,\beta)$ will be defined by quadrates, to within an additive constant. As anticipated, we will obtain a family of parallel surfaces satisfying the demand. On explicating equation (46), we obtain

$$\vec{P}'_\alpha\vec{\ell}'_\beta = \vec{P}'_\beta\vec{\ell}'_\alpha \quad .$$

Now

$$\vec{P}'_\alpha = k\vec{\ell} \ , \qquad\qquad \vec{\ell}\vec{\ell}'_\beta = 0 \quad ,$$

and eventually we obtain

$$\vec{P}'_\beta\vec{\ell}'_\alpha = 0 \quad . \tag{47}$$

$\vec{\ell}$ is the unit vector of the tangent tc the curve C at the point P when s is the arc of this curve

$$\vec{\ell}'_\alpha = \vec{\ell}'_s \frac{ds}{d\alpha} = \frac{\vec{n}}{R}\frac{ds}{d\alpha} \quad ,$$

where $\vec{n}$ is the unit vector of the principal normal. Eqaution (47) can be stated as

$$\vec{n}\,\vec{P}'_\beta = 0 \quad .$$

It follows that, for equation (47) to occur, it is necessary and sufficient that the osculating planes of the curves C are normal at each point P to Σ. These osculating planes are the tangent planes to the developlables formed by the tangents to the curves C. These are the focal tangent planes to the second sheet of the focal surface or to the focal curye if there is deçeneracy. The focal planes relative to the same line of the congruence are rectangular, and this suffices.

The theorem stated has therefore been proved in the case where the two sheets of the focal surface do not simultaneously reduce to curves. Lets look at this last situation. Let Λ and Λ' denote the two curves. The developables of the congruence are cones whose vertices P are on Λ and

which have Λ' as their directrix, and for the cones whose vertices P' are on Λ', the directrix is Λ. The condition of orthogonality implies, given that PT is the tangent to Λ at the point P, the plane PTP' is orthogonal to the tangent plane at the cone with vertex P along PP'. The cone of vertex P is a cone of revolution. The curve Λ' is common to five cones of the second degree; it is a conic. Λ and Λ' are two focal conics; the congruence is the congruence of normals of a cycloid of Dupin. The proposition is proved for all cases.

Remarks. I. The statement omits the case where the envisaged congruence would be a congruence of normals of a surface with umbilics forming two-dimensional domains. This is a case which only arises for the congruence of lines passing through a fixed point when we restrict our attention to analytic congruences. In this case, the congruence is the congruence of normals to concentric spheres.

II. The curves C of Σ, whose osculating planes are normal to Σ, are geodesics of Σ.

254. Applications. The theorem of Malus.

I. A given surface S can, in many ways, be a focal surface of a congruence of normals. It suffices to take on S, a sheaf of geodesics, the tangents to which form a congruence whose focal planes are rectangular. If S is a sphere, the geodesics are great circles.

II. An arbitrary surface is not in general the complete focal surface of a congruence of normals. For the congruence of the bitangent lines (which only exists when the degree is at least four, in the algebraic case) does not, in general, possess the property of orthogonality of the focal planes.

THE TRAJECTORIES OF LIGHT RAYS

We mentioned in no. 217 that following the Fermat principle, the trajectories of light rays within a medium of index of refraction $f(M)$, are the extremals of $\int f(M)\, ds$. When $f(M)$ is constant, these trajectories are lines. In the case when we pass from one medium of refraction index n_1 to another of index n_2 separated by a surface S, then the trajectory of minimum duration in going from A to B (fig. 83) is taken along a broken line AMB, such that

$$n_1 AM + n_2 MB$$

is a minimum, where M is on S.

If we give M a displacement $\delta\vec{M}$, then the variation of the above quality is

$$n_1 \vec{\ell}_1 \delta\vec{M} - n_2 \vec{\ell}_2 \delta\vec{M} \quad ,$$

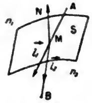

Fig. 83.

where $\vec{\ell}_1$ and $\vec{\ell}_2$ are unit vectors on AM and MB in the sense of A tending towards B. It must be zero, for any $\overrightarrow{\delta M}$ orthogonal to the unit vector $\vec{N}$ of the normal to S at the point M. Hence $n_1\vec{\ell}_1 - n_2\vec{\ell}_2$ must be along $\vec{N}$, or

$$n_1\vec{\ell}_1 - n_2\vec{\ell}_2 = k\vec{N} \quad . \tag{48}$$

This implies that $\vec{N}$, $\vec{\ell}_1$ and $\vec{\ell}_2$ must be coplanar and that if I_1 and I_2 are taken to be the angles of AM and MB with the normal $\vec{N}$, $n_1 \sin I_1 = n_2 \sin I_2$. These are the laws of Descartes.

In the case of a reflection of light rays on the surface S, we likewise obtain the law of Descartes by assuming that the index n_1 is replaced by $-n_1$ after the reflection. Formula (48) holds true on replacing n_2 by $-n_1$.

<u>The Theorem of Malus.</u> *When the rays of a congruence of normals undergo a reflection or a refraction, then they form a congruence of normals following this reflection or refraction.*

The congruence taking the refraction (the reflection reduces to a refraction on account of the above discussion) can be defined by

$$\vec{M} + \vec{\ell}_1 u \quad ,$$

where M is on the limit surface S. $\vec{M}$ depends on two parameters α and β, as does $\vec{\ell}_1$. In order for the congruence to be a congruence of normals, it is necessary and sufficient to be able to take for u, a function of α and β such that $d\vec{M} + \vec{\ell}_1 du + d\vec{\ell}_1 u$ is orthogonal to $\vec{\ell}_1$, hence that the product with $\vec{\ell}$ is zero; we must then have

$$du = -\vec{\ell}_1 \overrightarrow{dM} \quad .$$

It is necessary and sufficient for the second member of this equality to be a total differential. Now, on account of (48), we have

$$n_1\vec{\ell}_1 \overrightarrow{dM} = n_2\vec{\ell}_2 \overrightarrow{dM} \quad .$$

When the first member is an exact differential, then so too is the second member, and this proves the theorem.

255. Curves whose tangents belong to a line complex

A complex of curves is a collection of curves depending on 3 parameters. Through an arbitrary point of the space, there passes an infinity of curves. In particular, the lines

$$x = az + p, \qquad y = bz + q \tag{49}$$

form a complex when the coefficients a,b,p,q alone are subjected to satisfy a relation

$$F(a,b,p,q) = 0 \ . \tag{50}$$

The lines of the complex that pass through a point $M_0(x_0,y_0,z_0)$ are the generators of a cone known as *the cone of the complex.* Its equation is obtained by stating that the line $x - x_0 = a(z - z_0)$, $y - y_0 = b(z - z_0)$ belongs to the complex, which in turn gives

$$F\left[\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}, x_0 - \frac{x-x_0}{z-z_0} z_0, y_0 - \frac{y-y_0}{z-z_0} z_0\right] = 0 \ . \tag{51}$$

When F is a polynomial in a,b,p,q, the degree of the cone of the complex defines *the order of the complex.* The lines of the complex which are situated in a plane have as their envelope, a curve known as *the curve of the complex.* If $Ax + By + Cz + D = 0$ is the equation of the plane in question and $ux + vy + w = 0$ is the projection onto the plane Oxy of a line of this plane, then by stating that this line belongs to the complex, we obtain the tangential equation of the projection onto Cxy of the curve of the complex

$$F\left[\frac{Cv}{Bu-Av}, \frac{-Cu}{Bu-Av}, \frac{Dv-Bw}{Bu-Av}, \frac{Aw-Du}{Bu-Av}\right] = 0 \ .$$

In the algebraic case, we see that the class of this curve is equal to the order of the complex.

LINEAR COMPLEXES

The complexes of order one are said to be linear. The cone of such a complex is a plane. The plane corresponding to M will be called *the polar plane* of M. The class being 1, the lines of the complex situated in a plane pass through a fixed point of this plane, which we call the *pole* or *focus* of this plane.

A linear complex induces a correspondence between every point M and a plane which is the polar plane of M, and to every plane m, a point which is the pole of m. When a plane turns about a line Δ, the pole of this plane describes a line Δ'. In effect, if p_1 and p_2 are two positions of the plane, P_1 and P_2 their respective poles, and M an arbitrary point of Δ, then the lines MP_1 and MP_2 belong to the complex. If P is taken to

be arbitrary on the line Δ' joining P_1 and P_2, the plane $\Delta'M$ has M as its pole, hence the line MP belongs to the complex; as M is arbitrary, P is the pole of the plane ΔP. The lines Δ and Δ' are said to be *conjugate lines* with respect to the complex.

The equation of a linear complex is obtained by observing that, if p and q are given, then a and b must satisfy a linear relationship between a and b. Similarly, if a and b are given, p and q satisfy a linear relationship. The function F must be bilinear in a,b; p,q, and as equation (51) must represent a plane, we see that there are no terms in ap and in bq, and that $aq - bq$ is the cnly term of the second degree. The equation is of the form

$$Aa + Bb + Cp + Dq + E(aq - bp) + G = 0 ,$$

where $A,\dots,G$ are constants.

We can simplify this equation by means of a suitable choice of axes. If we take as the axis Oz, the conjugate line of the line at infinity of the plane, a plane taken to be the xy plane, then the polar plane of any point of Oz will be parallel to Oxy. Following equation (51), this plane is given by

$$Ax + By + Cz_0x - Dz_0y + G(z-z_0) = 0 .$$

We must have $A = B = C = D = 0$ and the equation of the complex is

$$aq - bp = H = \text{const.} \neq 0 . \tag{52}$$

The polar plane of the point x_0, y_0, z_0 has as its equation

$$y_0x - x_0y - H(z - z_0) = 0 . \tag{53}$$

CURVES WHOSE TANGENTS BELONG TO A LINE COMPLEX

By taking the complex in the reduced form (52), we see that the tangent at the point x_0, y_0, z_0 must be in the plane (53); we must then have

$$y_0dx_0 - x_0dy_0 - Hdz_0 = 0 .$$

These curves then satisfy the equation

$$ydx - xdy = Hdz . \tag{54}$$

They are defined when their projection onto Oxy is given. If $y = f(x)$ is the equation of this projection, we have

$$Hz = \int (f - xf')dx = 2\int f(x)dx - xf(x) .$$

We can take these curves to be of the form

$$y = y'(x) , \qquad z = \frac{1}{H}[2g(x) - xg'(x)] ,$$

where $g(x)$ is an arbitrary differentiable function, a primitive of $f(x)$. Following the equation in (54), we have at a point $M_0(x_0,y_0,z_0)$ of such a curve,

$$y_0 - xy_0' - Hz_0' = 0 , \qquad -x_0y_0'' - Hz_0'' = 0 .$$

The osculating plane then has as its equation

$$-y_0(x - x_0) + x_0(y - y_0) + H(z - z_0) = 0 ;$$

this is the plane given by (53). We thus obtain a theorem due to Appell: *when the tangents of a skew curve belong to a line complex, then the osculating plane at each point is the polar plane of this point.*

Remark. The reduced equation (52) was taken for any axes. But the conjugate lines of the lines of the plane at infinity, all pass through the polar of this plane; they are parallel. There exists therefore, a direction of parallel planes which are perpendicular to the line Δ, the locus of their foci. By taking one of these planes as the xy plane and Δ for Oz, with Ox and Oy rectangular, then we obtain the reduced equation (52) in rectangular axes.

RULED SURFACES BELONGING TO A LINE COMPLEX

When the generators of a ruled surface S belong to a line complex, then there exists on each generator G, two points P and Q at which the tangent plane to S is the polar plane of this point P, Q. For if π is a variable plane passing through G, its pole M is a point of G and the correspondence between M and π, is homographic (a bijective algebraic correspondence). The correspondence between M and the tangent plane Ω to S at a point M, is also homographic (the theorem of Chasles, no. 247). The correspondence between Ω and π is also homographic. There exists two points P and Q whose polar plane is the plane tangent to S; they correspond to the double elements of this homography. The points P and Q describe two curves Γ and Λ whose tangents belong to the complex. At P, the osculating plane of Λ is the polar plane of the complex following Appell's theorem; it is the tangent plane of S and the line Γ is an asympotic line; likewise for Λ. We know two asymptotic lines (non-rectilinear) of S; *the other asymptotic lines are determined by a single quadrature* (no. 243).

IV. MINIMAL ANALYTIC SURFACES

256. Minimal surfaces of revolution and minimal ruled surfaces

On account of the result of no. 201, the surface S of minimum area passing through a given closed curve C', satisfies the Lagrange equation

$$r(1+q^2) - 2pqs + t(1+p^2) = 0 \quad .$$

The first member is the expression occurring in the numerator of the mean curvature (no. 225). The minimal surfaces can only be obtained by means of surfaces of zero mean curvature. This result is due to Meusnier. Conversely, a surface S whose mean curvature is zero will satisfy the first order conditions if we can find the minimal surface for a contour C' traced on S. We are thus led to study those surfaces of zero mean curvature and *all* of these surfaces are known as minimal surfaces.

The Theorem of Meusnier. *The only minimal surfaces of revolution are the surfaces engendered by the rotation of a chain turning about the base.*

Since, having found the centers of principal curvature of a surface of revolution (no. 229), these points are the center of curvature of the meridian and the point of intersection of the normal to the surface with the axis of revolution (fig. 76, no. 229). If I and I' are taken to denote these centers of curvature corresponding to the point M of S, then the surface will be minimal when I and I' are symmetric with respect to M. The meridian Γ will be such that, if M is one of its points, I the center of curvature of Γ at the point M and I' the point of intersection of the normal to Γ at the point M with the axis of revolution Δ, then I and I' are symmetric with respect to M.
This is a property of the chains having Δ as their base; this a property which only belongs to these curves. We can see this by a direct method, which leads to a second order differential equation which can be easily integrated. Alternatively, we can see this without calculating; we deserve that the property assumes the form of a second order differential equation since it only involves the curvature and the length of the normal. There will only be two arbitrary constants and the chains in question depend on the constants. The theorem is thus proved.

Remark. The results of nos. 189 and 198 show that, in this case, the minimal surface indeed corresponds to a problem of minimization.

The Theorem of Meusnier - Catalan. *The only minimal ruled surfaces are the ruled helicoids with a planar direction.*

Meusnier (1776) proved that this helicoid is a minimal surface. Catalan (1842) proved that there are no other ruled minimal surfaces.

In order to prove this theorem, we rely on the fact that when the mean curvature is zero, the asympotic tangents are rectangular, and vice-versa. For a ruled surface, one of the systems of asymptotic lines is constituted by the generators. In order for the surface to be minimal, it is necessary and sufficient that the orthogonal trajectories of the generators are asymptotic lines, i.e. that the osculating plane of one such trajectory is tangent to the surface. The generators will therefore be the principal normals of their orthogonal trajectories. On account of Bertrand's theorem (no. 215), this is only possible when the orthogonal trajectories are circular helices. The ruled surface is engendered by the principal normals to a circular helix; this is a ruled helicoid with a planar direction. Q.E.D.

257. Minimal analytic surfaces. The representation in terms of lines of null length.

When a surface S is analytic, we can define on the surface, the lines of null length, namely lines along which the ds^2 is zero. On decomposing the ds^2 into a product of factors, as in no. 239, we see that these lines are defined by the differential equations

$$du + \frac{F \pm i\sqrt{EG - F^2}\ dv}{E} = 0 \quad .$$

In particular, when the surface S is a minimal surface, *then the lines of null-length of* S *form a conjugate network*. In order to see this, it is sufficient to show that the directions of these lines are conjugate. Now, if we take the point M of the surface as origin and the tangent plane as the plane Oxy, then the lines of null length are given by $dy \pm i dx = 0$. The indicatrix is an equilateral hyperbola, the directions $\pm i$ are conjugate with respect to this hyperbola.

Let us assume that we can relate the surface to the lines of null length, and call u and v the corresponding parameters. Since the coordinate lines are conjugate, we have, letting $\vec{M}(u,v)$ denote the vector which defines the surface,

$$\vec{M}''_{uv} = A(u,v)\vec{M}'_u + B(u,v)\vec{M}'_v \tag{55}$$

since $D' = 0$ (cf. no. 245). Moreover, since the coordinate lines are lines of null length

$$(\vec{M}'_u)^2 = 0 \ , \qquad (\vec{M}'_v)^2 = 0 \quad . \tag{56}$$

On differentiating these two equations with respect to v and u, respectively and applying (55), we obtain

$$A(u,v)\vec{M}'_u\vec{M}'_v = 0 ,$$

$$B(u,v)\vec{M}'_u\vec{M}'_v = 0 .$$

Now $\vec{M}'_u\vec{M}'_v$ is nonzero, otherwise on account of (56), we would have $ds^2 \equiv 0$. $A(u,v)$ and $B(u,v)$ are therefore identically zero, and condition (55) reduces to

$$\vec{M}''_{uv} = 0 .$$

The minimal surfaces are surfaces of translation. We have

$$\vec{M}(u,v) = \vec{P}(u) + \vec{Q}(v) , \tag{57}$$

with

$$\vec{P}'^2 = 0 , \qquad \vec{Q}'^2 = 0 . \tag{58}$$

Remark. We have implicitly assumed that the surface S is real. If S is imaginary, then the two systems of lines of null length can be identified. We shall put this case aside.

EXPLICIT FORMULAE

Monge and Legendre deduced equations (58) from the explicit formulae which had been modified by Enneper and Weierstrass. Let $f(u)$, $g(u)$, $h(u)$ be the components of $\vec{P}(u)$ in rectangular axes. The equation

$$f'(u)^2 + g'(u)^2 + h'(u)^2 = 0$$

can be written as

$$[f'(u)+ig'(u)][f'(u)-ig'(u)] = -h'(u)^2$$

and if we set

$$U = \frac{f'(u)+ig'(u)}{-h'(u)}$$

we have

$$f'(u) - ig'(u) = \frac{h'(u)}{U} ,$$

and consequently

$$\frac{f'(u)}{1-U^2} = \frac{g'(u)}{i(1+U^2)} = \frac{h'(u)}{2u} . \tag{59}$$

The case where U is constant, as studied by Poisson, would provide an imaginary cylinder. When U varies, u is a function of U and, on denoting by

$$\frac{F(U)dU}{2du}$$

the values of the ratios (59), we obtain

$$f(u) = \frac{1}{2}\int(1 - U^2)F(U)dU\ , \qquad g(u) = \frac{i}{2}\int(1 + U^2)F(U)dU\ ,$$

$$h(u) = \int UF(U)dU\ .$$

Likewise we may treat $\vec{Q}(v)$. By changing i to $-i$ and u and U to v and V respectively, we obtain the components

$$\frac{1}{2}\int(1 - V^2)G(V)dV\ , \qquad -\frac{i}{2}\int(1 + V^2)G(V)dV\ , \qquad \int VG(V)dV\ .$$

The coordinates of a point of the surface are

$$\left.\begin{aligned} x &= \frac{1}{2}\int(1 - U^2)F(U)dU + \frac{1}{2}\int(1 - V^2)G(V)dV\ , \\ y &= \frac{i}{2}\int(1 + U^2)F(U)dU - \frac{i}{2}\int(1 + V^2)G(V)dV\ , \\ z &= \ UF(U)dU + \int VG(V)dV\ . \end{aligned}\right\} \tag{60}$$

The functions $F(U)$ and $G(V)$ are arbitrary analytic functions. On replacing $F(U)$ by $H'''(U)$ and $G(V)$ by $K'''(V)$ and then proceeding to integrate, we obtain the explicit formulae, due to Weierstrass, without the signs of quadratures:

$$\left.\begin{aligned} x &= \frac{1-U^2}{2}H''(U) + UH'(U) - H(U) + \frac{1-V^2}{2}K''(V) + VK'(V) = K(V)\ ; \\ y &= i\left[\frac{1+U^2}{2}H''(U) - UH'(U) + H(U)\right] - i\left[\frac{1+V^2}{2}K''(V) - VK'(V) + K(V)\right]; \\ z &= UH''(U) - H'(U) + VK''(V) - K'(V)\ . \end{aligned}\right\} \tag{61}$$

The minimal algebraic surfaces are obtained by taking $H(U)$ and $K(V)$ to be algebraic functions in (61). It is clear that if $H(U)$ and $K(V)$ are algebraic functions then elimination of U and V from equations (61) and the equations satisfied by H and K will yield an algebraic relation between x, y, and z.

For the proof of the converse, we refer to Darboux's *Théorie générale des surfaces*, vol. I. In particular, on taking $H(U)$ to be a rational function, $K(U)$ to be a conjugate imaginary fraction, then on taking $U = u + iv$, $V = u - iv$, we obtain a unicursal real minimal surface. More generally, the real minimal surfaces are obtained by taking $F(U)$ and $G(U)$ in (60) to be imaginary conjugates and on calculating the integrals along the the conjugate imaginary paths in the planes of U and V (see the cited work of Darboux). Finally, we have

$$x = \mathcal{R}\int(1-U^2)F(U)dU, \qquad y = \mathcal{R}i\int(1+U^2)F(U)dU\ ,$$

$$z = \mathcal{R}\int 2UF(U)dU\quad .$$

We can also assign to each analytic function of the complex variable U, a real minimal surface defined to within a translation.

258. Asymptotic lines and lines of curvature. The conformal representation.

We shall assume that the surface is real. The formulae in (60) at once give the expressions of the partial derivatives of x,y,z with respect to U and V. We deduce that

$$ds^2 = (1+UV)^2F(U)G(V)dUdV\ .$$

The direction parameters of the normal are, following division by a common factor,

$$i(U+V)\ , \qquad U-V\ , \qquad i(UV-1)\ ,$$

where the sum of the squares of these numbers is $-(1+UV)^2$. For a unit normal vector we can take the vector with components

$$X = \frac{U+V}{1+UV}\ , \qquad Y = i\ \frac{V-U}{1+UV}\ , \qquad Z = \frac{UV-1}{1+UV}\ . \tag{62}$$

The second fundamental form is given by

$$\mathcal{L}du^2 + 2\mathfrak{M}dudv + \mathcal{N}dv^2 = \vec{N}d^2\vec{M}\quad .$$

We obtain

$$-F(U)dU^2 - G(V)dV^2\quad .$$

The asymptotic lines are obtained by quadratures

$$\int\sqrt{F(U)}\ dU = \pm\ i\int\sqrt{G(V)}\ dV\ .$$

The normal curvature is given by

$$\frac{1}{R_N} = -\frac{F(U)dU^2+G(V)dV^2}{(1+UV)^2F(U)G(V)dUdV}\quad . \tag{63}$$

The lines of curvature are given by the formulae of no. 227. Although the functions and variables here are complex, the result of the calculation is the same when these elements are real; we obtain this by stating the condition for an extremum of the second member of (63), which at once yields

$$F(U)dU^2 - G(V)dV^2 = 0\quad .$$

The lines of curvature are obtained by the same quadratures as those for the asymptotic lines; we have

$$\int\sqrt{F(U)}\ dU = \pm\int\sqrt{G(V)}\ dV\quad .$$

From this, the radii of principal curvature are deduced. We have

$$2R = \pm(1+UV)^2\sqrt{F(U)G(V)} \ .$$

THE SPHERICAL REPRESENTATION

The point of the sphere of unit radius which corresponds to the point M of the surface in the spherical representation (no. 230) is the point whose coordinates are given by (62). The square of arc element of this spherical representation is

$$d\sigma^2 = \frac{4dUdV}{(1+UV)^2} \ .$$

This is proportional to the ds^2 of the surface. *The spherical representation of a minimal surface is therefore a conformal representation.* (Bonnet's theorem). By virtue of the general results relating to the spherical representation, we may pass from the system of tangents to the curves at S at a point M, to the sheaf of the corresponding tangents of the spherical representation by making a symmetry with respect to a principal tangent.

Conversely, the fact that the spherical representation is conformal implies that the asymptotic tangents which are orthogonal to the corresponding tangents of the spherical representation are rectangular, hence the surface is a minimal surface.

<u>Enneper's Formulae. Bour's Theorem.</u> We may write

$$ds^2 = \frac{1}{2}\,(1+UV)^2\sqrt{F(U)G(V)}\sqrt{2F}\ dU\sqrt{2G}\ dV \ ,$$

$$d\sigma^2 = 2\,\frac{\sqrt{2F}\ dU\sqrt{2G}\ dV}{(1+UV)^2\sqrt{F(U)G(V)}} \ .$$

If we let

$$\alpha = \int\sqrt{2F(U)}\ dU \ , \qquad \beta = \int\sqrt{2G(V)}\ dV \ ,$$

and introduce the inverse functions

$$U = A(\alpha) \ , \qquad V = B(\alpha) \ ,$$

then the formulae in (60) and the expressions of ds^2 and $d\sigma^2$ take the following form

$$\left.\begin{aligned} x &= \int \frac{1-A^2}{4A'}\,d\alpha + \int \frac{1-B^2}{4B'}\,d\beta \quad , \\ y &= i\int \frac{1+A^2}{4A'}\,d\alpha - i\int \frac{1+B^2}{4B'}\,d\beta \quad , \\ z &= \int \frac{A}{2A'}\,d\alpha + \int \frac{B}{2B'}\,d\beta \quad ; \end{aligned}\right\} \tag{64}$$

$$ds^2 = \frac{(1+AB)^2 d\alpha d\beta}{4A'B'} \quad , \qquad d\sigma^2 = \frac{4A'B'}{(1+AB)^2}\,d\alpha d\beta \ . \tag{65}$$

The result of this is that when the $d\sigma^2$ of the sphere is stated in the form

$$d\sigma^2 = k(\alpha,\beta)d\alpha d\beta \ ,$$

it can then take the form of the second formula in (65). Consequently,

$$ds^2 = \frac{1}{k(\alpha,\beta)}\,d\alpha d\beta$$

will be the square of the line element of a minimal surface. *This is Bour's Theorem* (1862). The formulae (64) are *Enneper's formulae* (1964). It suffices to state $d\sigma^2$ in the form (65) to obtain A and B.

The lines of curvature will be given by

$$\alpha+\beta = \text{const.} \qquad \text{and} \qquad \alpha-\beta = \text{const.}$$

THE CONFORMAL REPRESENTATION OF RIEMANN

If at the point $M(U,V)$ of the surface (60) we assign the point P of the plane of rectangular coordinates X_1, Y_1 defined by

$$X_1 + iY_1 = \int\sqrt{2F(U)}\,dU \ , \qquad X_1 - iY_1 = \int\sqrt{2G(V)}\,dV \ ,$$

then the ds^2 of the surface will take the form

$$ds^2 = \frac{1}{2}\,(1+UV)^2\sqrt{FG}(dX_1^2 + dY_1^2) = R(dX_1^2 + dY_1^2) \ ,$$

where R is one of the radii of principal curvature. This form of the ds^2 shows that the planar representation so obtained, is conformal. The lines of curvature of the surface are represented by the lines $X_1 = \text{const.}$, $Y_1 = \text{const.}$, and the asymptotic lines by $X_1 + Y_1 = \text{const.}$, $X_1 - Y_1 = \text{const.}$ The system of parallels to the axes is isothermal (no. 239) and *the lines of curvature of a minimal surface form an isothermal system.*

259. Minimal surfaces whose lines of curvature are planar. Enneper surfaces.

If we assume that a family of lines of curvature of any surface is composed of plane curves, then the spherical representation of this family is composed of plane curves; the spherical representation of this family is composed of circles, since the tangents to one of these spherical curves are parallel to those of the line of curvature and, consequently, parallel to a plane which implies that the curve is planar.

For a minimal surface, the system of image lines of the lines of curvature on the sphere, is isothermal. It follows that, *if one of the systems of these image curves is formed by circles, then it is the same for the other system, and the circles of these two systems are the circles of two orthogonal sheaves on the sphere.* In order to prove this, we can via stereographic projection, reduce matters to the planar case, since this projection is a conformal representation and assigns an orthogonal isothermal system to an orthogonal isothermal system. It therefore amounts to proving that: *If one of the families of curves of a planar, orthogonal isothermal system is formed by circles, then these circles form a linear sheaf.*

The ds^2 of the plane with respect to this isothermal system can, in fact, be expressed in the form

$$ds^2 = \Omega(u,v)(du^2 + dv^2) ,$$

where the circles correspond to $u = \text{const.}$, for example (no. 239). If we put

$$X = u , \qquad Y = v ,$$

where X and Y are the rectangular coordinates of the plane, then we represent the system on the parallels to the axes conformally. The circles C give the lines D, $X = \text{const.}$

Let us apply the principle of symmetry (I, 216): if D' and D'' are symmetric with respect to D, then the corresponding circles C' and C'' are symmetric with respect to C, and hence form part of the same sheaf since the symmetry with respect to a circle is an inversion with respect to this circle. Now, if we consider C_0 and C_1 corresponding to D_0 and D_1, then the circle C_2 corresponding to the line D_2 equidistant from D_0 and D_1, forms part of the sheaf C_0, C_1. We might say that this circle is the circle equidistant from C_0 and C_1. Likewise, the circles C_3 and C_4 equidistant from C_0C_2 and C_2C_1 form part of this sheaf; and so on. The proof of the theorem follows by continuity.

Remark. The proof assumes nothing, *a priori*, about the domain in which the isothermal system is considered; this domain can only contain a part of the circles in question.

CONSEQUENCE

A minimal surface whose lines of curvature in one of the systems, are planar, has its two systems of lines of curvature composed of plane curves and the spherical images of these lines form two sheaves of orthogonal circles on the sphere.

ENNEPER SURFACES

These correspond to the case where the sheaves of circles are circles tangent at a point to the same line and the orthogonal circles. The sphere is defined by the coordinates (62) of one of its points (the curves $U = \text{const.}$ and $V = \text{const.}$ are the generators of the sphere). If we consider the circular sections through the planes $-\lambda(Z-1) = X$ and $-\mu(Z-1) = Y$, where λ and μ are parameters, then we obtain the relations

$$2\lambda = U + V , \qquad 2\mu = i(V - U) .$$

Consequently, on taking $F(U) = k$, $G(V) = k$, where k is a constant, we obtain a surface whose lines of curvature have as their images on the sphere, the circles which just happened to have been considered. This is the Enneper surface. We can take k to be a numerical constant; all of the other surfaces in question will be deduced by similitude from the one that we will so define. For $k = 3$, we will have, on account of the formulae appearing towards the end of no. 257

$$x = \mathbb{R}(3U - U^3), \qquad y = \mathbb{R}i(3U + U^2), \qquad z = \mathbb{R}3U^2 ,$$

and, on setting $U = u - iv$, we obtain

$$x = 3u + 3uv^2 - u^3, \qquad y = 3v + 3u^2v - v^3, \qquad z = 3u^2 - 3v^2 .$$

This is a unicursal surface. On stating that a point of the surface is on a plane $\delta x + \delta' y + \delta'' z + \delta''' = 0$, we obtain a relation between u and v which is of degree three in u or v; a line will thus be intersected in nine points. The surface is of degree nine.

The planes of the lines of curvature, lines corresponding to $u = \text{const.}$ and $v = \text{const.}$ and which are of degree three, are the planes with equations

$$x + uz = 3u + 2u^3, \qquad y - vz = 3v + 2v^3 .$$

We have

$$ds^2 = 9(1 + u^2 + v^2)^2(du^2 + dv^2) ;$$

the arc of the lines of curvature are expressed rationally as functions of u and v. The asymptotic lines have as their equations $u + v = \text{const.}$ and $u - v = \text{const.}$, these are cubic (skew) curves whose arc is tabulated in terms of rational functions. The components of the unit vector of the normal are

$$X = \frac{2u}{1+u^2+v^2}, \qquad Y = \frac{-2v}{1+u^2+v^2}, \qquad Z = \frac{u^2+v^2-1}{1+u^2+v^2} .$$

When $u = v + c$, $c = \text{const.}$, we have

$$X + Y + cZ = c ,$$

and the asymptotic lines are helices. The radii of principal curvature are given by

$$2R = \pm\ 3(1 + u^2 + v^2)^2 \ ;$$

the surface of centers is unicursal. The curves Γ on which R is constant correspond to $r^2 = u^2 + v^2 = \text{const.}$ and they are rectifiable. The geodesics are given by the extremals of

$$\int (1 + u^2 + v^2) \sqrt{du^2+dv^2} \ .$$

If we take polar coordinates $u = r\cos\theta$, $y = r\sin\theta$, we can consider

$$\int (1 + r^2) \sqrt{1+r^2\theta'^2} \ dr \quad ;$$

the geodesics are defined by

$$(1 + r^2) \frac{r^2\theta'}{\sqrt{1+r^2\theta'^2}} = c = \text{const.}$$

For $c = 0$, we have $\theta = \text{const.}$, the orthogonal trajectories of the curves Γ defined above are geodesics, and the lines Γ are geodesically equidistant. Moreover, in polar coordinates, the ds^2 takes the geodesic form. The other geodesics have as their equation

$$\theta = \int \frac{cdr}{r\sqrt{r^2(1+r^2)^2-c^2}}$$

on taking r^2 as the variable. In general, one has to introduce elliptic functions.

THE GEOMETRIC DEFINITION

The planes $x = 0$ and $y = 0$ are planes of symmetry of the surface (change u to $-u$ and v to $-v$). The planes of the lines of curvature $v = \text{const.}$ intersect the plane $x = 0$ to which they are orthogonal, along the lines $y - vz = 3v + 2v^3$, which are normals of the parabola $y = 4v$, $z = -2v^2 + 1$. These planes π are therefore the normal planes of the parabola

$$x = 0 \ , \qquad y = 4v \ , \qquad z = -2v^2 + 1 \ . \tag{P}$$

Likewise, the planes π' of the lines of curvature $u = \text{const.}$ are the normal planes to the parabola

$$x = 4u \ , \qquad y = 0 \ , \qquad z = +2u^2 - 1 \ . \tag{P'}$$

The segment joining these two incidence points on P and P' has as its parameters $4u$, $-4v$, $2u^2 + 2v^2 - 2$. It is parallel to the normal at the point $M(u,v)$ of the surface. The center of this segment has

$$2u, \quad 2v, \quad (u^2 - v^2) ;$$

it is in the tangent plane to the surface at the point M, a plane whose equation is

$$2ux - 2vy + (u^2 + v^2 - 1)z + 3v^2 - 3u^2 + v^4 - u^4 = 0 \quad .$$

The parabolas P and P' are focal to each other. Consequently: *The Enneper surface is the envelope of the mediating planes of the segments* QQ' *obtained by joining two arbitrarily close points* Q,Q' *taken on two focal parabola* P *and* P'. *The lines of curvature passing through the point of contact of the mediating plane of* QQ' *with the surface, are in the normal planes to* P *at the point* Q *and to* P' *at the point* Q'.

260. A minimal surface passing through a given contour

The problem concerned with passing a surface whose area is minimum through a given closed curve, as posed by Lagrange, was solved for certain polygonal contours by Riemann, Weierstrass and Schwartz. Plateau observed that, with respect to the theory of capillarity, we can physically realize a surface which is theoretically a minimal surface passing through a given contour, by affectively constructing this contour out of a length of wire and immersing it in a solution of glycerine. On turning the contour, we ascertain that a liquid pellicle forming a surface and passing through the contour, is obtained (1873). Although this experiment cannot imply in any way the possibility of solving the Lagrange problem, we habitually refer to this problem as the *Plateau problem.*

The simultaneous use of the conformal representations considered in no. 258 allows us, in certain cases, to determine the functions F and G appearing in (60). For a discussion of this method, we refer to volume 1 of *Darboux's Théorie des Surfaces*, wherein an exposition of the results of Riemann, Weierstrass and Schwarz, is also to be found. The Plateau problem relative to a polygonal contour amounts to studying a linear second order differential equation

$$\frac{d^2\theta}{du^2} + p\,\frac{d\theta}{du} + q = 0 \quad ,$$

in which the coefficients p and q are rational fractions with unknown coefficients. This is a problem somewhat similar to the problem of conformal representation of a polygonal domain on a half-plane, but more complicated to solve. This method, which solves the problem only for certain polygonal contours, was taken up by R. Garnier (1926-1928) who obtained a solution in the general case of polygonal contours and, by a limiting process, in the case of contours formed by a finite number of arcs of curves whose curvature is bounded.

An entirely different method adopted by J. Douglas gave rise to the complete solution of the problem under the sole hypothesis that the given contour is a simple continuous closed curve (without multiple points). We refer, on this subject, to the original paper of Douglas, 'Solution of the problem of Plateau' (Transactions of the American Mathematical Society, vol. XXXIII, 1931).

V. FUNDAMENTAL PROBLEMS IN THE THEORY OF SURFACES

261. The Codazzi formulae

If we take the formulaes (39) of no. 230

$$\vec{N}'_u = m\vec{M}'_u + n\vec{M}'_v \,, \qquad \vec{N}'_v = m'\vec{M}'_u + n'\vec{M}'_v \,,$$

where m, n, m', n' are given by the formulae (40) of no. 230, then on differentiating with respect to v and u and equating the values obtained, we obtain

$$(m-n')\vec{M}''_{uv} + n\vec{M}''_{vv} - m'\vec{M}''_{uu} + \vec{M}'_u\left(\frac{\partial m}{\partial v} - \frac{\partial m'}{\partial u}\right) + \vec{M}'_v\left(\frac{\partial n}{\partial v} - \frac{\partial n'}{\partial u}\right) = 0 \,.$$

If we now multiply by $\vec{M}'_u$ then by $\vec{M}'_v$ and refer to the results of no. 226, we see that

$$\frac{m-n'}{2}\frac{\partial E}{\partial v} + n\left(\frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u}\right) - \frac{m'}{2}\frac{\partial E}{\partial u} + E\left(\frac{\partial m}{\partial v} - \frac{\partial m'}{\partial u}\right) + F\left(\frac{\partial n}{\partial v} - \frac{\partial n'}{\partial u}\right) = 0 \,,$$

$$\frac{m-n'}{2}\frac{\partial G}{\partial u} + \frac{n}{2}\frac{\partial G}{\partial v} - m'\left(\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}\right) + F\left(\frac{\partial m}{\partial v} - \frac{\partial m'}{\partial u}\right) + G\left(\frac{\partial n}{\partial v} - \frac{\partial n'}{\partial u}\right) = 0 \,.$$

On replacing in these formulae, the quantities m, n, m', n' by their values extracted from the equations (40) of no. 230, but introduce D, D', D'' in place of $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$, we obtain identities of the following form

$$H^2\left[\frac{\partial D'}{\partial u} - \frac{\partial D}{\partial v}\right] = D\left[\frac{1}{2}F\frac{\partial G}{\partial u} + F\frac{\partial F}{\partial v} - \frac{1}{2}E\frac{\partial G}{\partial v} - G\frac{\partial E}{\partial v}\right] + D'\left[F\frac{\partial E}{\partial v} - 2F\frac{\partial F}{\partial u} + G\frac{\partial E}{\partial u}\right] + D''\left[E\frac{\partial F}{\partial u} - \frac{1}{2}E\frac{\partial E}{\partial v} - \frac{1}{2}F\frac{\partial E}{\partial u}\right];$$

$$H^2\left[\frac{\partial D'}{\partial u} - \frac{\partial D''}{\partial u}\right] = D\left[G\frac{\partial F}{\partial v} - \frac{1}{2}G\frac{\partial G}{\partial u} - \frac{1}{2}F\frac{\partial G}{\partial v}\right] + D'\left[F\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial v} + E\frac{\partial G}{\partial v}\right] + D''\left[\frac{1}{2}F\frac{\partial E}{\partial v} + F\frac{\partial F}{\partial u} - \frac{1}{2}G\frac{\partial E}{\partial u} - E\frac{\partial G}{\partial u}\right].$$

These are the Codazzi formulae. We see that D, D' and D" satisfy these two equations, to which we must add the Gauss formula relating to the total curvature (no. 226), which can be written as

$$DD''-D'^2 = \begin{vmatrix} E & F & \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial v} \\ \frac{1}{2}\frac{\partial E}{\partial u} & \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} & \tau' \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}\frac{\partial E}{\partial v} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{1}{2}\frac{\partial E}{\partial v} & \frac{1}{2}\frac{\partial G}{\partial u} & 0 \end{vmatrix}$$

where

$$\tau' = \frac{\partial^2 F}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 G}{\partial u^2} - \frac{1}{2}\frac{\partial^2 E}{\partial v^2} .$$

262. The fundamental problems in the theory of surfaces

The relations between the coefficients of the two fundamental quadratic forms which has just been stated, are the only relations between these coefficients. One can show, in fact, that if they are satisfied by a system of functions E, F, G, D, D', D" of u and v such that $EG - F^2 = H^2 \neq 0$, and which satisfy differentiability conditions, then there exists a surface, defined to within a translation and symmetry, for which E, F, G, D, D' and D" are the coefficients of the two fundamental forms.

When E, F and G alone are given, then the surface is defined up to a deformation. We can in general deform a surface in such a way that a given curve described on this surface happens to coincide with a curve previously given.

These problems for which we refer the reader to the above cited work of Darboux, happen to relate to the theory of partial differential equations.

Chapter XV

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

A first order partial differential equation is an equation between an unknown function z of several variables, these variables and the first partial derivatives of z with respect to these variables. The equation is linear if the partial derivatives occur to the first degree.

The integration of the linear equations was undertaken by Lagrange (1779); the theory of multipliers which was discussed in no. 158, is due to Jacobi (1842). Systems of linear equations were studied by Jacobi and Clebsch (1862-1866), and then by Lie. The singular integrals of linear equations were studied by Goursat (1889); the geometric interpretation of first order partial differential equations, is due to Monge (1795). The works of Pfaff and Poisson date from 1814 and 1809.

Euler (1770) had integrated some nonlinear equations in two variables, but the general theory of these equations is credited to Lagrange (1772) and Charpit (1784); the integration of equations with more than two variables was reduced by Pfaff (1814) to the integration of a differential system. Cauchy introduced his technique in 1819. The method of Jacobi for equations of more than two variables and the work of Hamilton on canonical equations, date from 1836 and 1834 respectively. Equations with two variables were studied in greater depth by Bonnet (1857), du Bois-Reymond (1864), Lie (1872) and Darboux (1883).

In the exposition which follows, we will dwell, above all, on the case of equations in two variables; for the general case, we refer to the works of Goursat and E. Cartan mentioned in no. 283. The existence theorems of the solutions will be reduced to those concerning differential equations by applying the theorem of Poincaré-Weierstrass on initial conditions.

I. LINEAR EQUATIONS

263. The general integral

We have seen in no. 156 that given a partial differential equation of the form

$$X_1 \frac{\partial z}{\partial x_1} + X_2 \frac{\partial z}{\partial x_2} + \cdots + X_n \frac{\partial z}{\partial x_n} = 0 \quad , \tag{1}$$

where $X_1, X_2, \ldots, X_n$ are functions of $x_1, x_2, \ldots, x_n$ admitting continuous first order partial derivatives in a domain Δ, and z an unknown function of $x_1, x_2, \ldots, x_n$, then the determination of the solutions of this equation reduces to the integration of the differential system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \quad . \tag{2}$$

If $F_1(x_1,\ldots,x_n),\ldots,F_{n-1}(x_1,\ldots,x_n)$ is a system of $(n-1)$ first integrals of this system (2) whose functional determinant with respect to $n-1$ of the variables $x_1,x_2,\ldots,x_n$, is nonzero in a region Δ' of Δ and if

$$\Phi(F_1,F_2,\ldots,F_{n-1})$$

is a function of $F_1,F_2,\ldots,F_{n-1}$ admitting continuous first partial derivatives, then this function $z = \Phi$ satisfies equation (1). Conversely, *the solutions of (1) admitting continuous first partial derivatives are of the form* (no. 156).

Remark. If $X_1,X_2,\ldots,X_n$ are analytic functions in Δ, then $F_1,F_2,\ldots,F_{n-1}$ are analytic (nos. 155, 47). We will obtain analytic solutions of (1) by taking an analytic function of $F_1,F_2,\ldots,F_{n-1}$ for Φ and vice-versa. Therefore, by taking Φ be a nonanalytic function, we will obtain a non-analytic solution of equation (1).

THE GENERAL LINEAR EQUATION

An equation with partial derivatives of the first order, and linear, is an equation of the form

$$Y_1 \frac{\partial z}{\partial x_1} + Y_2 \frac{\partial z}{\partial x_2} + \cdots + Y_n \frac{\partial z}{\partial x_n} - Z = 0 \quad , \tag{3}$$

where z is an unknown function of $x_1,x_2,\ldots,x_n$ and $Y_1,Y_2,\ldots,Y_n,Z$ are functions of $x_1,x_2,\ldots,x_n$ z which we shall assume to be defined in a domain $\Delta(x_1,x_2,\ldots,x_n,z)$ where they admit continuous first partial derivatives.

Let us consider a solution taken in the implicit form

$$V(x_1,x_2,\ldots,x_n,z) = 0 \quad , \tag{4}$$

where the function V has first partial derivatives, nonzero when taken with respect to z. We will have

$$\frac{\partial V}{\partial x_j} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial x_j} = 0 \ , \qquad j = 1,2,\ldots,n \ , \tag{5}$$

and, for equation (3) to be satisfied, it is necessary and sufficient to have

$$Y_1 \frac{\partial V}{\partial x_1} + Y_2 \frac{\partial V}{\partial x_2} + \cdots + Y_n \frac{\partial V}{\partial x_n} + Z \frac{\partial V}{\partial z} = 0 \tag{6}$$

when z is replaced by its value deduced from equation (4). But if we consider a solution of equation (6) (an equation which is of type (1)), $V(x_1,\ldots,x_n,z)$ say, and if we equate this function to zero, then the function $z(x_1,x_2,\ldots,x_n)$ so obtained will have partial derivatives which are given by the equations in (5) and which will satisfy equation (3), since (6) will be satisfied provided $\frac{\partial V}{\partial z}$ is not zero. *We thus obtain the solutions to equation (3) by equating to zero the general solution of equation (6).* We intend to show that the general solution of equation (3) can be obtained by this method. Let $G_1,G_2,\ldots,G_n$ be n first integrals of equation (6) providing a fundamental solution. We have

$$Y_1 \frac{\partial G_k}{\partial x_1} + Y_2 \frac{\partial G_k}{\partial x_2} + \cdots + Y_n \frac{\partial G_k}{\partial x_n} + Z \frac{\partial G_k}{\partial z} = 0, \qquad k = 1,2,\ldots,n\,. \tag{7}$$

Let $z(x_1,x_2,\ldots,x_n)$ be a solution of equation (3). If, in $G_k(x_1,\ldots,x_n,z)$, we replace the variable z by $z(x_1,\ldots,x_n)$, then we obtain a function $g_k(x_1,x_2,\ldots,x_n)$. The functional determinant of these functions is

$$\frac{D(g_1,\ldots,g_n)}{D(x_1,\ldots,x_n)} = \left\| \frac{\partial G_k}{\partial x_1} + \frac{\partial G_k}{\partial z}\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial G_k}{\partial x_n} + \frac{\partial G_k}{\partial z}\frac{\partial z}{\partial x_n} \right\|$$

or

$$\frac{D(G_1,\ldots,G_n)}{D(x_1,\ldots,x_n)} + \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} \cdot \frac{D(G_1,\ldots,G_k,\ldots,G_n)}{D(x_1,\ldots,x_{j-1},z,x_{j+1},\ldots,x_n)} \quad .$$

Now, if we consider the system of equations (7) as equations in $Y_1,\ldots,Y_n$, we can solve it with respect to these quantities since, by hypothesis, the determinant is nonzero. On introducing the values deduced from equation (3), we obtain

$$\begin{vmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_n} & Z\frac{\partial G_1}{\partial z} \\ \cdot & \cdots & \cdot & \cdot \\ \frac{\partial G_n}{\partial x_1} & \cdots & \frac{\partial G_n}{\partial x_n} & Z\frac{\partial G_n}{\partial z} \\ \frac{\partial z}{\partial x_1} & \cdots & \frac{\partial z}{\partial x_n} & -Z \end{vmatrix} = 0 \quad ,$$

i.e.,

$$Z\,\frac{D(g_1,\ldots,g_n)}{D(x_1,\ldots,x_n)} = 0 \quad .$$

Consequently, either the functional determinant of $g_1,g_2,\ldots,g_n$ with respect to $x_1,\ldots,x_n$ is zero, or else $Z=0$. In this last case, the solution of the system in (7) reduces to $Y_1 = 0,\ldots,Y_n = 0$. The condition

$$\frac{D(g_1,\ldots,g_n)}{D(x_1,\ldots,x_n)} = 0$$

implies (I, 126) that, at least in a bounded domain, there exists a relation between $g_1,g_2,\ldots,g_n$. We thus arrive at the following statement:
A general integral of equation (3), where the Y_j and Z admit continuous first partial derivatives in a domain Δ, satisfies an equation obtained by equating to zero a function of n distinct first integrals of the associated differential system

$$\frac{dx_1}{Y_1} = \frac{dx_2}{Y_2} = \cdots = \frac{dx_n}{Y_n} = \frac{dz}{Z} \quad .$$

Besides the solutions so defined, equation (3) can be satisfied by a function z *satisfying the conditions* $Y_j = 0$, $j = 1,2,\ldots,n$ and $Z=0$.

The solution of the equation therefore reduces to the solution of an implicit equation

$$V(G_1,G_2,\ldots,G_n) = 0 \quad .$$

V must be taken to be a function such that it is possible to deduce from the equation a function $z(x_1,\ldots,x_n)$ admitting partial derivatives.

264. The method of Cauchy. Characteristics.

In order to facilitate the geometric interpretation, we shall confine ourselves to an equation defining a function of two variables. This can be carried through to the general case.

Consider an equation

$$P_p + Q_q - R = 0 \;, \qquad p = \frac{\partial z}{\partial x} \;, \qquad q = \frac{\partial z}{\partial y} \;, \tag{8}$$

where P,Q,R are functions of x,y,z. We look for solutions $z=\phi(x,y)$. It is possible to consider the geometric represetnation of these solutions by surfaces; in particular, we look for the integral surfaces. Equation (8) can be interpreted in the following way: the tangent plane at the point $M(x,y)$ of an integral surface has as its equation

$$Z - z = p(X-x) + q(Y-y) \quad ,$$

where X,Y,Z are variable coordinates. Condition (8) indicates that this

plane contains the line with equations

$$\frac{X-x}{P} = \frac{Y-y}{Q} = \frac{Z-z}{R} \,. \tag{9}$$

We now consider a domain $\Delta(x,y,z)$ in which the functions P,Q,R are uniform and continuous. *At each point* M *of* Δ *we shall consider the line defined by (9), or, which is the same, the vector* $\overrightarrow{MV}$ *with components* P,Q,R. *We wish to obtain surfaces* S, *such that at each point* M *of a surface* S, *the tangent plane contains the vector* $\overrightarrow{MV}$.

CHARACTERISTICS

If S is an integral surface, $z=\phi(x,y)$, then at each point M of S, the tangent plane contains the vector $\overrightarrow{MV}$. Let $m(x,y)$ be the projection of the point $M(x,y,\phi)$ onto the plane Oxy. The vector $\overrightarrow{MV}$ projects along a vector $\overrightarrow{mv}$ with components $P(x,y,\phi),Q(x,y,\phi)$. Under the sole condition that P and $Q(x,y,z)$ satisfy the Lipschitz conditions with respect to y and z, P is nonzero at any point of the domain δ in question, and that S admits a tangent plane which varies continuously, then the differential equation

$$\frac{dx}{P(x,y,\phi)} = \frac{dy}{Q(x,y,\phi)} \tag{10}$$

admits solutions (no. 151). There exist tangent curves to the vector $\overrightarrow{mv}$ at each of their points m; on the surface S, there correspond to these curves, the curves Γ tangent to the corresponding vector $\overrightarrow{MV}$ at each of their points M. Since on S we have $dz = pdx+qdy$, then the ratios (10) equate to

$$\frac{dz}{pP(x,y,\phi)+qQ(x,y,\phi)} \,,$$

and the denominator of this expression is equal to $R(x,y,\phi)$ since S is an integral surface of equation (8). Consequently, along Γ, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \,, \tag{11}$$

and Γ is an integral curve of this differential system. *It does not depend on the surface* S *in question.* The curves Γ can be defined, *a priori*, as solutions of the system in (11). Every integral surface is a locus of such curves. Conversely, every surface locus of curves Γ that admits a tangent plane is an integral surface.

The curves Γ *which are solutions of the system in (11), existing under the indicated conditions* (no. 151), *are called characteristic curves,* or in short, simply *characteristics.* This terminology is due to Monge.

An integral surface S, *given a priori, can be regarded as the locus of characteristics which intersect a given curve* Λ, *noncharacteristic, situated on* S.

Remark. We also see that it is possible to find integrals of the partial differential equation in (8) by just assuming that P,Q,R are continuous, satisfy the Lipschitz conditions in y and z for example, and that P is nonzero in Δ.

THE CHARACTERISTIC CONGRUENCE

If we assume that P,Q,R have continuous first derivatives, $P \neq 0$, then the system in (11) admits first integrals; the characteristics can be written in the form

$$u(x,y,z) = a , \qquad v(x,y,z) = b , \tag{12}$$

where u and v are differentiable functions, a and b arbitrary constants and

$$\frac{D(u,v)}{D(y,z)} \neq 0 \quad .$$

The congruence (12) is the *characteristic congruence.* The integral surfaces are the *surfaces of the congruence;* they are obtained on stating an arbitrary relation $\Phi(a,b) = 0$ between a and b, where Φ is always taken to be such that z can be deduced as a differentiable function.

In practice, we will obtain the general integral of equation (8) by seeking two distinct first integrals of the system in (11), the functions $u(x,y,z)$ *and* $v(x,y,z)$ *say, and on writing*

$$\Phi(u,v) = 0 \quad ,$$

where Φ *is an arbitrary function.*

Remark. We recover the result of no. 263: the hypothesis with $P \neq 0$, discards solutions of the sort $P = Q = R = 0$.

265. The Cauchy Problem. Singularities.

The Cauchy problem consists finding an integral surface passing through a given curve. We assume this curve to be continuous, simple and admitting a continuous tangent. Following the above discussion, such a surface can only be uniquely determined when the curve Λ is not a characteristic. *If* Λ *is not a characteristic, then there exists a unique integral surface passing through* Λ*, this being the locus of the characteristics* Γ *which intersect* Λ. The Cauchy problem is thus solved when the given curve Λ is taken to be in a domain Δ in which the method of integration applies and when Λ is not a characteristic. It is worthwhile noting that, in such a domain Δ, the characteristics which can be taken to be of the form in (12), do not have an envelope; the curve Λ can only be tangent to particular characteristics. In general, the characteristic passing through a point M of Λ will not be tangent to Λ.

If the curve Λ is given in the parametric form

$$x = f(t) , \qquad y = g(t) , \qquad z = h(t) ,$$

and if the characteristics Γ are taken to be of the form in (12), then we can assert that Γ intersects Λ by showing that the values of t defined by

$$u(f(t),g(t),h(t)) = a ; \qquad v(f(t),g(t),h(t)) = b \tag{13}$$

are the same. By eliminating t from these two equations, we obtain the relation $\Phi(a,b) = 0$ which will define the integral surface in question.

When the curve Λ is the intersection of two surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$, then we obtain the relation $\Phi(a,b) = 0$ by eliminating x,y,z from these two surface equations and the equations (12).

THE ANALYTIC CASE

When P,Q,R are analytic functions, the characteristics are analytic. If the given curve Λ is itself analytic, then the surface passing through Λ will be analytic. In effect, the functions u,v,f,g,h in (13) are analytic, hence the first members are analytic. The value of t obtained from one of these equations will be analytic (we take t to be in the neighborhood of a value for which Γ and Λ are not tangential), and on introducing this value into the second equation, we obtain an analytic relation between a and b. Finally, by replacing a and b in this relation by analytic functions u and v, we produce an analytic function of x,y,z.

THE PARTIAL DIFFERENTIAL EQUATION OF A CONGRUENCE OF CURVES

Given a congruence of curves of the form (12), the differential equations of those curves which are derived from the equations $u'_x dx + u'_y dy + u'_z dz = 0$ and $v'_x dx + v'_y dy + v'_z dz = 0$, are

$$\frac{dx}{\frac{D(u,v)}{D(y,z)}} = \frac{dy}{\frac{D(u,v)}{D(z,x)}} = \frac{dz}{\frac{D(u,v)}{D(x,y)}} .$$

The curves of the congruence are then characteristics of the partial differential equations

$$p\,\frac{D(u,v)}{D(y,z)} + q\,\frac{D(u,v)}{D(z,x)} - \frac{D(u,v)}{D(x,y)} = 0 .$$

This is the partial differential equation of the surfaces belonging to the congruence, or in short, *the partial differential equation of the congruence*.

If the congruence is given in the general form

$$U(x,y,z,a,b) = 0 , \qquad V(x,y,z,a,b) = 0 , \tag{14}$$

we then recover the preceding case by solving in terms of a and b. But it is also possible to state directly the partial differential equation of the congruence by eliminating dx, dy, dz from the equations

$$U'_x dx + U'_y dy + U'_z dz = 0$$

$$V'_x dx + V'_y dy + V'_z dz = 0$$

$$dz = p dx + q dy \quad ,$$

which state that the tangent plane to a surface of the congruence passes through the tangent at this point to the curve of the congruence, and further eliminate a and b from equations (14) and the equation

$$\begin{vmatrix} U'_x & U'_y & U'_z \\ V'_x & V'_y & V'_z \\ p & q & -1 \end{vmatrix} = 0 \tag{15}$$

thus obtained.

In the case where U and V are polynomials in x,y,z,a,b, algebraic elimination will provide a partial differential equation of the form

$$F(x,y,z,p,q) = 0$$

where F is a polynomial in x,y,z,p,q. This equation will decompose into a product of equations of the form (8) since it is also possible to make the elimination by solving the equations (14) in terms of a and b and then substituting in the expression in (15). We may, in general, a single equation so obtained from those in (8), whose coefficients P,Q,R will be multiform functions.

SINGULARITIES AND SINGULAR INTEGRALS

More generally, if P,Q,R are taken to be multiform functions, then the differential system in (11), in certain elementary cases, may at the very least be replaced (after raising to the appropriate powers) by a system of the form

$$A\left(x,y,z\ \frac{dy}{dx}\ ,\ \frac{dz}{dx}\right) = 0\ , \qquad B\left(x,y,z\ \frac{dy}{dx}\ ,\ \frac{dz}{dx}\right) = 0\ , \tag{16}$$

to which the discussion in no. 149 is relevant. The congruence of these characteristic curves may admit a focal surface; the characteristics will be tangent to the singular integral curves of the system situated on this focal surface. In this case, the focal surface will be an integral surface of the partial differential equation in (8), to which the other integral surfaces will be tangential, with the exception of those formed by the characteristics admitting an envelope situated on this focal surface. *This will be a singular integral.*

The Cauchy problem will be indeterminate when we are given a curve Λ situated on the focal surface; through such a curve there will pass two integral surfaces, in general: the singular integral, which is the focal surface, and an ordinary integral surface which is tangent to it.

Remark. On account of the result of no. 149, the singular integral will not exist in general: the surface on which the system of equations in (16) is non-solvable (the functional determinant of A and B with respect to dy/dx and dz/dx is equal to zero) is a locus of turning points of the curves of the characteristic congruence.

266. Examples

I. The partial differential equation of the cones whose vertex is the origin, is $z - px - qy = 0$; the equations of the characteristics are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \quad .$$

These yield the first integrals $\frac{y}{x}$, $\frac{z}{x}$ and $\Phi(\frac{y}{x}, \frac{z}{x}) = 0$ determines the integral surfaces.

II. The surfaces of revolution about Oz (rectangular axes) are characteristics by virtue of the fact that the normal to any point of the surface intersects Oz, which gives the condition $py - qx = 0$. The characteristics are the solutions of the equations

$$-\frac{dx}{y} = \frac{dy}{x}, \qquad dz = 0 \quad ,$$

which has $x^2 + y^2$ and z as first integrals. The general integral is $\Phi(z, x^2 + y^2) = 0$. The characteristics are the circles with Oz as axis.

We remark that, more generally, the equations of the certain type in (1), do enter in the general theory.

III. The conoids with direction plane Oxy, whose generators intersect Oz, are characterized by the fact that the tangent plane at the point x,y,z intersects Oz at a point denoted by z. In effect, this condition yields the equation $px + qy = 0$, whose characteristics are the lines $z = \text{const.}$, $\frac{y}{x} = \text{const.}$, and the general integral is $\Phi(z, \frac{y}{x}) = 0$.

IV. Orthogonal trajectories of a one parameter family of surfaces. If $f(x,y,z) = a$ is the equation of the given surfaces S, where a is the parameter, then the surfaces which intersect the surfaces S at a right angle will be determined by the condition that at the arbitrary point $M(x,y,z)$, the tangent planes are orthogonal. Taking f'_x, f'_y, f'_z to be the parameters of the normals to the surface S and $p,q,-1$ for the surface in question, we obtain (with respect to rectangular axes)

$$pf'_x + qf'_y - f'_z = 0 \quad .$$

The characteristics defined by

$$\frac{dx}{f'_x} = \frac{dy}{f'_y} = \frac{dz}{f'_z} \quad ,$$

are the orthogonal trajectory curves of the surfaces S. If $u(x,y,z) = a$ and $v(x,y,z) = b$, are the equations of this congruence of curves, then the orthogonal surfaces will be given by $\Phi[u(x,y,z), v(x,y,z)] = 0$, where Φ is an arbitrary function.

V. Let us consider the surfaces S which possess the following property: given a point M of S, the tangent plane at M intersects the plane passing through M and perpendicular to a given line Δ, along a line MT whose distance from Δ is equal to a given number a. If we take Δ as the axis Oz, $Oxyz$ tri-rectangular, then the property is encoded within the equation

$$(px + qy)^2 - a^2(p^2 + q^2) = 0 \quad .$$

On expanding and solving for p/q, we obtain the linear equation with non-uniform coefficients

$$p(x^2 - a^2) + q\left(xy \pm a\sqrt{x^2 + y^2 - a^2}\right) = 0 \quad .$$

The characteristics are given by $dz = 0$ and the equation

$$\frac{dx}{x^2 - a^2} = \frac{dy}{xy \pm a\sqrt{x^2 + y^2 - a^2}} \quad ,$$

which can also be written as

$$(xdy - ydx)^2 - a^2(dx^2 + dy^2) = 0 \quad .$$

This is a Clairault equation

$$xy' - y = \pm a\sqrt{1 + y'^2} \quad .$$

The characteristics are given by

$$z = \lambda \ , \qquad x\mu - y = \pm a\sqrt{1 + \mu^2} \quad , \tag{17}$$

where μ and λ are arbitrary constants. But the differential system also admits the singular integrals

$$z = \lambda \ , \qquad x^2 + y^2 = a^2 \quad .$$

The nonsingular characteristics are the tangents Γ to the circles admitting Oz as their axis; the singular characteristics are these circles. This is

evident from their geometric definition. The integral surfaces are the ruled surfaces engendered by the tangents Γ. Their equations can be written collectively, on replacing μ by an arbitrary function of z in (17), by

$$(x\phi(z) - y)^2 - a^2(1 + (\phi(z)))^2 = 0 \quad , \tag{18}$$

where $\phi(z)$ is an arbitrary function. There is a singular integral which is the cylinder $x^2 + y^2 = a^2$.

Remark. Besides the integral surfaces which were just discussed in Example V and were given by equation (18), there exist integral surfaces which can be defined as follows: in the plane $z = \lambda$, we take an arc PQ of the circle and consider the tangents to the circle (fig. 84) at P and Q, where the actual positions of P and Q depend on λ which is varied. We obtain a surface whose tangent plane varies continuously when the positions of P and Q do likewise with λ, and which is composed of two pieces of ruled surfaces bounded by their lines of striction and a cylindrical portion. We thus obtain an integral surface.

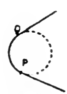

Fig. 84.

We may proceed likewise whenever the congruence of characteristics admits a focal surface.

We may compare the considerations with those developed in no. 63.

II. TOTAL DIFFERENTIAL EQUATIONS

267. A completely integrable equation. The existence and calculation of solutions.

A total differential equation is an equation of the form

$$dz = A_1 dx_1 + A_2 dx_2 + \cdots + A_n dx_n \quad , \tag{19}$$

where the A_j are functions of $x_1, x_2, \ldots, x_n, z$. We wish to determine z as a function of $x_1, x_2, \ldots, x_n$ in such a way that the second member becomes the differential of this function $z = \phi(x_1, \ldots, x_n)$ when z is replaced by $\phi(x_1, \ldots, x_n)$. The equation becomes equivalent to the system of partial differential equations

$$\frac{\partial z}{\partial x_j} = A_j(x_1, x_2, \ldots, x_n, z) \ , \qquad j = 1, 2, \ldots, n \quad . \tag{20}$$

Firstly, we shall consider the case of an equation in two variables:

$$dz = A(x,y,z)dx + B(x,y,z)dy \tag{21}$$

which is equivalent to

$$\frac{\partial z}{\partial x} = A(x,y,z) , \qquad \frac{\partial z}{\partial y} = B(x,y,z) . \tag{22}$$

We shall assume that A and B are functions of x,y,z defined in a domain Δ which they have continuous first partial derivatives. If $z = \phi(x,y)$ is a solution, then it admits partial derivatives with respect to x, and then y, or to y and then x; these are continuous on account of (22) and are therefore equal, such that for $z = \phi(x,y)$, we have

$$\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} B = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial z} A . \tag{23}$$

We say that equation (21) [*or (22)*] *is completely integrable when equation (23) becomes an identity for any* x,y,z *in* Δ.

When the equation is not completely integrable, its solutions $z = \phi(x,y)$ can only be solutions of equation (23) when the latter is regarded as an equation in z. In general, there will only be a finite number of solutions (this could in fact be zero). For example, when we take $A \equiv yz$, $B \equiv z$, there will be a single solution $z = 0$; when $A \equiv y$, $B \equiv 2x$, there are no solutions.

A COMPLETELY INTEGRABLE EQUATION

If equation (21) is completely integrable in Δ, *there exists a unique solution* $z = \phi(x,y)$ *which takes for* $x = x_0$, $y = y_0$ *a value* $z_0(x_0,y_0z_0$ in $\Delta)$ *and is defined within a neighborhood of this point.*

If, in the first place, we assign to y the value y_0 in the first equation in (22), then we obtain a differential equation

$$\frac{dz}{dx} = A(x,y_0,z)$$

admitting a unique solution $z = \psi(x)$ which for $x = x_0$ takes the value z_0, providing $x - x_0$ is sufficiently small; $\psi(x)$ admits a continuous derivative. Let us proceed to integrate the second equation in (22), where x is a constant, and take as the initial conditions $z = \psi(x)$ for $y = y_0$. We obtain a solution $z = \phi(x,y)$ which is a function of y and of the parameter x and which admits a continuous derivative with respect to x. We have

$$\frac{\partial \phi}{\partial y} = B(x,y,\phi(x,y)) , \tag{24}$$

On referring to the proofs based on the Picard method (nos. 151-154), we see that $\partial\phi/\partial x$ also has a continuous derivative with respect to y; this function of y satisfies the equation obtained by differentiating (24) with respect to x (an equation with variations)

$$\frac{d}{dy}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial B}{\partial x}(x,y,\phi(x,y)) + \frac{\partial B}{\partial z}(x,y,\phi(x,y))\frac{\partial\phi}{\partial x}. \qquad (25)$$

Equation (25), in which $\phi(x,y)$ is known, defines $\frac{\partial\phi}{\partial x} = u(y)$. This function u is the solution of the equation which for $y = y_0$, takes the value $\psi'(x) = A(x,y_0,\psi(x))$. Now, on account of (23), equation (25) admits the solution $A(x,y,\phi(x,y))$ since the first member then becomes

$$\frac{\partial A}{\partial y}(x,y,\phi(x,y)) + \frac{\partial A}{\partial z}(x,y,\phi(x,y))\frac{\partial\phi}{\partial y},$$

where the last term is given by the equation (24). For $y = y_0$, these solutions are equal by the uniqueness theorem. We have

$$\frac{\partial\phi}{\partial x} \equiv A(x,y,\phi(x,y)),$$

which shows that $z = \phi(x,y)$ is (for $x - x_0$ and $y - y_0$ sufficiently small) is a solution of equation (21). By the construction of $\phi(x,y)$, this solution is unique in a square of center x_0,y_0. The proposition as stated is thus proved.

Remarks. I. We can extend the solution obtained as long as the differential equations introduced are extendable.

II. The uniqueness of the solution was only proved in the case of solutions defined in a domain containing the point (x_0,y_0). Let us consider a solution $z(x,y)$ which will be defined in a domain with the point (x_0,y_0) as a boundary point and which tends towards z_0 when the point (x,y) tends towards (x_0,y_0) along a certain line Γ. At the point (x_1,y_1), $z(x,y)$ will take a value z_1 and will coincide with the solution $z = \phi(x,y,x_1,y_1)$, about the point (x_1,y_1), as calculated by the above method; this will be a solution which will be defined within a square containing (x_0,y_0) providing that $|x - x_0| + |y_1 - y_0|$ is sufficiently small. The extension of $\phi(x,y,x_1,y_1)$ along Γ will coincide consistently with that of $z(x,y)$ and consequently, the value of $\phi(x_0,y_0,x_1,y_1)$ will be z_0; the function $\phi(x,y,x_1,y_1)$ will be the function $\phi(x,y)$. *The uniqueness is general.*

III. We can interchange the role of x and y i.e., integrate firstly with $x = x_0$ and then secondly with y constant.

IV. If A and B have continuous derivatives up to order q, then likewise for the solution obtained.

A PRACTICAL METHOD

The technique applied yields the solution which takes a given value z_0 at a given point x_0, y_0. This solution depends on an arbitrary constant z_0. We can obtain the solution directly as a function of an arbitrary constant and proceed as follows.

Let us take x to be constant in the second equation (22) and integrate the ordinary differential equation

$$\frac{dz}{dy} = B(x,y,z) \tag{26}$$

thus obtained. The solution will depend on a parameter x and on an arbitrary constant which could vary with x; this will be of the form $z = \Phi(x,y,\psi(x))$. The solutions of equation (21), whose existence is verified by the previous discussion, are necessarily of this form; moreover, the function Φ has a derivative with respect to the third variable ψ. Matters are reduced to determining $\psi(x)$ such that equation (22) is satisfied; by assuming $\psi(x)$ differentiable, we will need to have

$$\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial\psi}\frac{d\psi}{dx} = A(x,y,\Phi(x,y,\psi)) \quad .$$

The function ψ is thus given by a differential equation which can be written as

$$\frac{d\psi}{dx} = \frac{A(x,y,\Phi) - \dfrac{\partial\Phi}{\partial x}}{\dfrac{\partial\Phi}{\partial\psi}} \quad . \tag{27}$$

The existence theorem allows us to assert that the second member of this equation only depends on x; *ψ will be determined up to a constant.* But we can show directly that the second member of (27) does not depend on y when the equation is completely integrable. The numerator of the derivative of the second member with respect to y is

$$\left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial x\partial y}\right)\frac{\partial\Phi}{\partial\psi} - \left(A - \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial y\partial\psi} ,$$

and we have

$$\frac{\partial\Phi}{\partial y} = B(x,y,\Phi) , \qquad \frac{\partial^2\Phi}{\partial y\partial\psi} = \frac{\partial B}{\partial z}\frac{\partial\Phi}{\partial\psi} ,$$

$$\frac{\partial^2\Phi}{\partial x\partial y} = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}\frac{\partial\Phi}{\partial x} \quad .$$

We can see that this expression reduces to

$$\left[\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}B - \left(\frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}A\right)\right]\frac{\partial\Phi}{\partial\psi} \equiv 0 \quad .$$

Thus, we can integrate equation (26), then determine the function $\psi(x)$ *(which is the constant of integration) by integrating equation (27). The general solution will depend on an arbitrary constant.*

THE GENERAL CASE

The methods described above may be extended by recurrence. Let us confine ourselves to the 3-variable case. Let

$$du = A(x,y,z,u)dz + B(x,y,z,u)dy + C(x,y,z,u)dz \quad .$$

The equation will be completely integrable when we have, identically,

$$\frac{\partial A}{\partial y} + \frac{\partial A}{\partial u} B \equiv \frac{\partial B}{\partial x} + \frac{\partial B}{\partial u} A ,$$

$$\frac{\partial A}{\partial z} + \frac{\partial A}{\partial u} C \equiv \frac{\partial C}{\partial x} + \frac{\partial C}{\partial u} A ,$$

$$\frac{\partial B}{\partial z} + \frac{\partial B}{\partial u} C \equiv \frac{\partial C}{\partial y} + \frac{\partial C}{\partial u} B .$$

Under the conditions, the equations

$$\frac{\partial u}{\partial x} = A(x,y,z_0,u) , \qquad \frac{\partial u}{\partial y} = B(x,y,z_0,u) ,$$

are above the above type where z_0 is considered as one parameter; they admit a unique solution $u = \psi(x,y)$ which takes for $x = x_0$, $y = y_0$ (and $z = z_0$) a value u_0. The equation

$$\frac{\partial u}{\partial z} = C(x,y,z,u)$$

where x and y are considered as parameters, admits a unique solution which takes the value $\psi(x,y)$ for $z = z_0$. We can thus define a unique function $u = \phi(x,y,z)$ having derivatives with respect to x and y and if we assume that A,B,C have continuous first partial. These derivatives satisfy variational equations, for example

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial C}{\partial x}(x,y,z,\phi) + \frac{\partial C}{\partial u}(x,y,z,\phi)\,\frac{\partial \phi}{\partial x} \quad ,$$

and, as we saw above, this linear differential equation (for x and y constants) $\frac{\partial \phi}{\partial x}$ admits the solution $A(x,y,z,\phi)$, by virtue of the conditions of complete integrability. Since these solutions coincide for $z = z_0$, they are identical. Likewise we may obtain

$$\frac{\partial \phi}{\partial y} \equiv B(x,y,z,\phi)$$

and the above technique extends just as before.

268. <u>The geometric interpretation. Mayer's method. The analytic case.</u>

Let us consider an equation of the form (21) or (22) to describe the geometric interpretation in ordinary space. In looking for a solution we effectively look for a surface S situated in the domain Δ and such that at each of its points $M(x,y,z)$, the tangent plane coincides with a given plane, namely the plane with equation $Z = z + A(x,y,z)(X - x) + B(x,y,z)(Y - y)$.

Following the result obtained, the problem only involves solutions passing through an arbitrary point when the equation is completely integrable. If this integrability condition is satisfied, then there passes an integral surface through an arbitrary point in Δ.

APPLICATION

Consider a curve congruence defined by the equations

$$u(x,y,z) = a , \qquad v(x,y,z) = b ,$$

where u and v are given functions of x,y,z and a and b are arbitrary constants.

In general, there does not exist surfaces depending on one parameter whose congruence curves are orthogonal trajectories. Since, if p and q are the partial derivatives of $z = \phi(x,y)$ denoting a point, then for such a surface we must have (with respect to rectangular axes)

$$p \frac{\partial u}{\partial x} + q \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} = 0$$

$$p \frac{\partial v}{\partial x} + q \frac{\partial v}{\partial y} - \frac{\partial v}{\partial z} = 0 .$$

These equations express p and q at each point of the space as quotients of functional determinants. This is necessary for the system so obtained to be completely integrable.

If the congruence is taken to be of the form $U(x,y,z,a,b) = 0$, $V(x,y,z,a,b) = 0$, then similarly, we will have the conditions

$$pU'_x + qU'_y - U'_z = 0 ,$$

$$pV'_x + qV'_y - V'_z = 0 ,$$

deduced in stating that the normal to the surface in question is tangent to the congruence curve passing through this point. When the integrability condition is satisfied, then the corresponding total partial differential equation can be written by eliminating a and b from the four equations deduced; they will be in a nonsolvable form. In the first place, we can state the result of eliminating a and b from the four equations, thus obtaining the two equations

$$F(x,y,z,p,q) = 0 , \qquad G(x,y,z,p,q) = 0 .$$

We shall see later in no. 271 how the integrability condition can be proved to be satisfied without solving this system of two equations in p and q.

<u>Mayer's method</u>. Let us consider equation (21)

$$dz = A(x,y,z)dz + B(x,y,z)dy ,$$

where A and B satisfy the conditions of complete integrability along with

the conditions imposed in no. 267, which imply the existence of solutions. In order to obtain the integral surface which passes through the point $M_0(x_0,y_0,z_0)$, we consider the sections of this surface cut out by the planes passing through M_0 and parallel to Oz. We will obtain a curve in each of these planes and the locus of these curves will be the integral surface. In the plane $y - y_0 = m(x - x_0)$, where m is an arbitrary constant, we have $dy = mdx$,

$$dz = Adx + mBdx = [A(x,y_0 + m(x - x_0),z) + mB(x,y_0 + m(x - x_0),z)]dx \ .$$

The section of the surface through this plane will be given by integrating the differential equation thus obtained; we need to consider the solution which for $x = x_0$, takes the value z_0. The integral surface is seen to be of the form

$$z = \Phi(x,m,z^0), \qquad y = y_0 + m(x - x_0) \ ,$$

and on replacing m in the first equation by its value drawn from the second, the calculation is complete.

Mayer's method only requires a single integration of the differential equation.

Remark. More generally, we could employ a system of curvilinear coordinates $x = g(u,v)$, $y = h(u,v)$, such that the curves $v = \text{const.}$ are closed curves encircling the point (x_0,y_0) and reduce to this point for $v = 0$. Along the curves $u = \text{const.}$, from $v = 0$ to v, we integrate the equation

$$\frac{dz}{dv} = A[g(u,v),h(u,v),z]g'_v + B[g(u,v),h(u,v),z] \tag{28}$$

and we will obtain the coordinates of a point of the integral surface as functions of u,v. In particular, if we take polar coordinates, with pole (x_0,y_0), we obtain a minor modification of Mayer's method.

We ought to remark that equation (28) only satisfies the general conditions for the unique solution to be provided by the existence theorem when the curves $u = \text{const.}$ admit tangents at the point (x_0,y_0). We cannot consider the case of spirals.

THE ANALYTIC CASE

If the functions $A(x,y,z)$ and $B(x,y,z)$ are analytic about the point (x_0,y_0,z_0), then the solution provided by the technique of no. 267, is analytic. Since the function $\psi(x)$ obtained in the integration of the first differential equation, is analytic. The function $\phi(x,y)$ is therefore analytic, since for $B(x,y,z)$ analytic, the solution of the second differential equation is analytic with respect to y and $\psi(x)$.

269. Equations in a symmetric form. Bertrand's method.

Equation (21) can be written in the symmetric form

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \ , \tag{29}$$

where the variables x,y,z figure as before. After solving in terms of dz, where z is a function of x and y, the condition for complete integrability may be written as

$$\frac{\partial}{\partial y}\left(\frac{P}{R}\right) - \frac{\partial}{\partial x}\left(\frac{Q}{R}\right) \equiv 0 \ ,$$

or

$$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)R - P\frac{\partial R}{\partial y} + Q\frac{\partial R}{\partial x} + R\left(-\frac{\partial P}{\partial z}\frac{Q}{R} + \frac{\partial Q}{\partial z} - \frac{P}{R}\right) + P\frac{\partial R}{\partial z}\frac{Q}{R} - Q\frac{\partial R}{\partial z}\frac{P}{R} \equiv 0 \ .$$

After some simplification, we obtain

$$R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \equiv 0 \ . \tag{30}$$

If this condition is satisfied, then equation (29) can be integrated by taking as the unknown function, any one of the variables x,y,z which will be regarded as a function of the other two, since condition (30) is symmetric with respect to x,y,z,P,Q,R. The integral surfaces can then be written in the form $F(x,y,z) = \text{const.}$; here P,Q,R are direction parameters of the normal at a point of one such surface. We have

$$\frac{F'_x}{P} = \frac{F'_y}{Q} = \frac{F'_z}{R} \ .$$

Now, letting $\mu(x,y,z)$ denote the common values of these ratios, we see that we do have

$$\mu(Pdx + Qdy + Rdz) = dF \ . \tag{31}$$

μ *is an integrating factor of the first member of equation* (29). A completely integrable equation therefore admits integrating factors; μ is one of them and the functions $\mu\Phi(F)$, where Φ is some differentiable function, are the others. *Condition* (30) *is therefore a sufficient condition for the expression in* (29) *to admit an integrating factor. This condition is also necessary.* Since, if μ is an integrating factor, we must have

$$\frac{\partial \mu P}{\partial y} \equiv \frac{\partial \mu Q}{\partial x} \ , \qquad \frac{\partial \mu Q}{\partial z} \equiv \frac{\partial \mu R}{\partial y} \ , \qquad \frac{\partial \mu R}{\partial x} \equiv \frac{\partial \mu P}{\partial z} \ .$$

On expansion, we obtain equations of the form

$$P\frac{\partial \mu}{\partial y} - Q\frac{\partial u}{\partial x} = \mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ldots \ , \tag{32}$$

and we see that by multiplying these equations by R,P,Q respectively and then adding, that condition (30) is indeed necessary.

Every integrating factor ν can be written in the form $\mu g(x,y,z)$, and following the conditions in (32) (where μ is an integrating factor) we have

$$P\frac{\partial g}{\partial y} - Q\frac{\partial g}{\partial x} = 0 , \qquad Q\frac{\partial g}{\partial z} - R\frac{\partial g}{\partial y} = 0 ,$$

$$R\frac{\partial g}{\partial x} - P\frac{\partial g}{\partial z} = 0 ,$$

which shows that g is a function of F defined by (31). Every integrating factor is of the form $\mu\Phi(F)$.

VECTORIAL NOTATION

If we denote by $\overrightarrow{V(M)}$ the vector with components P,Q,R, where M is the point x,y,z, then finding an integrating factor for the first member of (29) entails finding a scalar $\mu(M)$ such that the vector product $\mu(M)\overrightarrow{V(M)}$ is a gradient. Condition (30) can be written as

$$\overrightarrow{V(M)}\ \mathrm{rot}\ \overrightarrow{V(M)} = 0 \quad ; \tag{33}$$

it can be obtained by stating that $\mu\vec{V}$ is a gradient. Effectively, for this to be so, it is necessary and sufficient that the rotational of this vector is zero (I,151), hence that

$$\overrightarrow{\mathrm{rot}\ \mu V} = 0 \quad . \tag{34}$$

Now on checking components, we see that

$$\overrightarrow{\mathrm{rot}\ \mu V} = \mu\ \overrightarrow{\mathrm{rot}\ V} + \overrightarrow{\mathrm{grad}\ \mu} \wedge \vec{V} \quad ,$$

and on scalar multiplication by $\vec{V}$, we see that equation (34) is equivalent to (33).

In the form (33), the integrability condition is invariant under a coordinate transformation, this being an intrinsic property. More generally, the expression in (31) shows that the property is still true under a point transformation.

A curve whose tangent at each point M is described by the vortex or rotational $\mathrm{rot}\ \overrightarrow{V(M)}$ is a *vortex line* and a *vortex surface* is a surface engendered by vortex lines. When the total differential equation, written as

$$\overrightarrow{V(M)}\ \overrightarrow{dM} = 0 \quad , \tag{35}$$

is completely integrable, then the vortex at each point M is orthogonal to $\overrightarrow{V(M)}$ [condition (33)]; the integral surfaces are then vortex surfaces, but the converse is not true in general.

THE RELATION WITH STOKES FORMULA

By Stokes formula, if C is taken to be a closed curve on a vortex surface, on which the normal vector is orthogonal to the rotational, then we have

$$\int_C \overrightarrow{V(M)}\,\overrightarrow{dM} = \iint_S \overrightarrow{\text{rot } V(M)}\,\vec{n}\,d\sigma = 0$$

(I,151). Conversely, if the first member of this equation is zero for every closed curve on a surface S, then on S, we have

$$\vec{n}\,\overrightarrow{\text{rot } V(M)} = 0 \quad ,$$

where S is a vortex surface.

If equation (35) satisfies the integrability condition (33), then we can determine the integral surfaces by determining the orthogonal vortex surfaces at each of their points M to the vector $\overrightarrow{V(M)}$. Given a point M_0, there passes through this point curves along which $\overrightarrow{V(M)}\,\overrightarrow{dM} = 0$; we can determine such a curve Γ by subjecting it to be in a given plane passing through M_0. There passes through Γ a vortex surface S which is determined by the vortex lines supported by Γ (we assume that Γ itself is not a vortex line). The surface S is an integral surface of (35). Effectively, if C is an arc of a curve described on S, joining two points M' and M'' on S, then we can form a closed line on S by considering the vortex lines T' and T'' which pass through M' and M'' bounded at these points and at the points M_1' and M_2'' where they intersect Γ, and by taking Γ from M_1 to M_1'', T', T'', C. The integral

$$\int \mu(M)\,\overrightarrow{V(M)}\,\overrightarrow{dM}$$

is zero on Γ from the definition of Γ; it is zero on T' and T'' since on these lines, $\overrightarrow{dM}$ is equal to $k(M)$ rot $\overrightarrow{V(M)}$ and (33) is satisfied. Since this integral is zero on the closed contour, it is zero on C. As C is arbitrary, it follows that throughout the surface S, equation (35) is satisfied (otherwise, there would exist a small curve arc C, on which the first member of (35) when divided by ds, the differential of the arc, would have a predetermined sign, and this would lead to a contradiction).

BERTRAND'S METHOD

Initially, this consists of integrating the partial differential equation

$$\overrightarrow{\text{rot } V(M)}\,\overrightarrow{\text{grad } f} = 0 \quad ,$$

or

$$\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\frac{\partial f}{\partial x} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\frac{\partial f}{\partial y} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{\partial f}{\partial z} = 0 \quad ;$$

the integral surfaces are vortex surfaces engendered by the vortex lines those being the solutions of the adjoint differential equation

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \quad . \tag{36}$$

Amongst the integral surfaces, we look for those for which equation (35) is satisfied. From what we have just said, it suffices to look for those curves on which (35) holds and to solve the Cauchy problem for these curves. Following Bertrand, we can, on the other hand, proceed as follows. If $u(x,y,z)$ and $v(x,y,z)$ are two distinct first integrals of the system in (36), then we have

$$\overrightarrow{\text{rot } V(M)}\,\overrightarrow{\text{grad } u} = 0 , \qquad \overrightarrow{\text{rot } V(M)}\,\overrightarrow{\text{grad } v} = 0 ,$$

$$\overrightarrow{\text{rot } V(M)}\,\overrightarrow{V(M)} = 0 \quad .$$

It follows that the vectors $\overrightarrow{\text{grad } u}$, $\overrightarrow{\text{grad } v}$, $\overrightarrow{V(M)}$ are coplanar; we have

$$\overrightarrow{V(M)} = \lambda\, \overrightarrow{\text{grad } u} + \mu\, \overrightarrow{\text{grad } v} \quad ,$$

where λ and μ are scalars and the total differential equation (35) becomes

$$\overrightarrow{V(M)}\,\overrightarrow{dM} = \lambda du + \mu dv = 0 \quad ,$$

where λ and μ are functions of x,y,z. But these can be expressed as functions of u,v and z for example. Since the equation is completely integrable, we will have the condition

$$\lambda \frac{d\mu}{dz} - \mu \frac{d\lambda}{dz} \equiv 0 \quad ,$$

showing that $\frac{\lambda}{\mu}$ only depends on u and v. Matters are reduced to integrating a differential equation and the integral surfaces will be obtained in the form $\Phi(u,v) = \text{const.}$

Remark. The introduction of the system in (36) implies, from our point of view, the existence hypothesis and continuity of the partial derivatives of P,Q,R up to the second order.

270. Examples

I. Consider the equation

$$p = \frac{2x + z^2 - x^2 - 2y^2}{2z} , \qquad q = \frac{4y + z^2 - x^2 - 2y^2}{2z} \quad ,$$

which is completely integrable. We need to integrate the differential equation $2zdz = (2x + z^2 - x^2 - 2y^2)dx$ $(y = \text{const.})$, which is linear when z^2 is taken to be the unknown function. We obtain

$$z^2 = x^2 + 2y^2 + \psi(y)e^x$$

and $\psi(y)$ is given by $\psi'(y) = \psi(y)$; we have $\psi(y) = Ce^y$ and

$$z^2 = Ce^{x+y} + x^2 + 2y^2 \quad ,$$

where C is an arbitrary constant.

II. Consider the equation

$$dz = \frac{(x^2+2x(y+z)-y^2-z^2)dx + (y^2+2y(x+z)-x^2-z^2)dy}{x^2+y^2-2z(x+y)-z^2} \quad .$$

The variables x and y play a symmetric role. The method of Mayer entails setting $y = mx$ and leads to a homogeneous differential equation. We set $z = tx$, which yields

$$x\frac{dt}{dx} + t = \frac{1+2m-m^2+2t-t^2+m(m^2+2m-1)+2tm^2-t^2m}{1+m^2-2t(1+m)-t^2} ,$$

then

$$\frac{dx}{x} = \frac{-t^2-2(1+m)t+1+m^2}{t^3+(1+m)t^2+(m^2+1)t+(m^2+1)(1+m)} dt \quad .$$

The denominator is put into the form

$$(t^2 + m^2 + 1)(t + m + 1)$$

and integration yields

$$x = C\frac{t+m+1}{t^2+m^2+1} \quad ,$$

where C is an arbitrary constant. The integral surfaces are spheres (with respect to rectangular axes)

$$x^2 + y^2 + z^2 = C(x + y + z) \quad .$$

If we write the equation in the form $Pdx + Qdy + Rdx = 0$, then we obtain

$$P = x^2 + 2x(y + z) - y^2 - z^2 , \qquad Q = y^2 + 2y(z + x) - z^2 - x^2 ,$$

$$R = z^2 + 2z(x + y) - x^2 - y^2 ;$$

the rotational has components $4(z - y)$, $4(x - z)$, $4(y - x)$. The vortex lines are given by

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} ;$$

we obtain the first integrals $x + y + z$, $x^2 + y^2 + z^2$, and hence the vortex surfaces $x^2 + y^2 + z^2 = \Phi(x + y + z)$. Applying Bertrand's method, the equation can be written as

$$\lambda d(x+y+z) + \mu d(x^2+y^2+z^2) = 0$$

with $\lambda = -(x^2+y^2+z^2)$, $\mu = x+y+z$, and the above result is recovered.

271. The case of nonsolvable equations

Let us assume that the total differential equation is of the form

$$F(x,y,z,p,q) = 0 , \qquad G(x,y,z,p,q) = 0 , \quad p = \frac{\partial z}{\partial x} , \quad q = \frac{\partial z}{\partial y} ,$$

for which the system constituted by these equations is solvable for p and q (with nonzero functional determinant). The complete integrability condition is

$$\frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} q \equiv \frac{\partial q}{\partial x} + \frac{\partial q}{\partial z} p \quad , \tag{37}$$

where p and q are taken to be functions of x,y,z. The partial derivatives which appear in this relation as calculated as functions of x,y,z,p,q by the usual method. For example, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \quad ,$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial G}{\partial q}\frac{\partial q}{\partial x} = 0 \quad ,$$

which gives

$$\frac{\partial q}{\partial x} = \frac{\dfrac{D(F,G)}{D(x,p)}}{\dfrac{D(F,G)}{D(p,q)}} \quad .$$

Likewise, we have

$$\frac{\partial p}{\partial y} = \frac{\dfrac{D(F,G)}{D(q,y)}}{\dfrac{D(F,G)}{D(p,q)}} , \qquad \frac{\partial p}{\partial z} = \frac{\dfrac{D(F,G)}{D(q,z)}}{\dfrac{D(F,G)}{D(p,q)}} , \qquad \frac{\partial q}{\partial z} = \frac{\dfrac{D(F,G)}{D(z,p)}}{\dfrac{D(F,G)}{D(p,q)}} .$$

The identity in (37) becomes

$$\frac{D(F,G)}{D(q,y)} + \frac{D(F,G)}{D(q,z)} q \equiv \frac{D(F,G)}{D(x,p)} + \frac{D(F,G)}{D(z,p)} p$$

where p *and* q *are to be replaced by functions of* x,y,z. *This identity must be a consequence of the identities* $F=0$, $G=0$. For example, it can be written as

$$\frac{\partial F}{\partial p}\left(\frac{\partial G}{\partial x} + p\,\frac{\partial G}{\partial z}\right) + \frac{\partial F}{\partial q}\left(\frac{\partial G}{\partial y} + q\,\frac{\partial G}{\partial z}\right) - \frac{\partial G}{\partial q}\left(\frac{\partial F}{\partial y} + q\,\frac{\partial F}{\partial z}\right) - \frac{\partial G}{\partial p}\left(\frac{\partial F}{\partial x} + p\,\frac{\partial F}{\partial z}\right) = 0.$$

We denote the first member by $[F,G]$, the bracket of F and G. By considering z as a function of x and y and p and q as independent

variables, it can be written as

$$[F,G] = \frac{\partial F}{\partial p}\frac{dG}{dx} + \frac{\partial F}{\partial q}\frac{dG}{dy} - \frac{\partial G}{\partial p}\frac{dF}{dx} - \frac{\partial G}{\partial q}\frac{dF}{dy} = 0 \quad . \tag{38}$$

In the case where F and G do not depend on z, the bracket expression simplifies: the d are replaced by the ∂ in (38) and the Poisson notation (F,G) is used in place of the Jacobi bracket $[F,G]$.

A PARTICULAR CASE

If $[F,G] \equiv 0$ for any x,y,z,p,q, then the system $F=a$, $G=b$ will be completely integrable for all constants a and b.

III. THE GENERAL FIRST ORDER EQUATION IN TWO VARIABLES

272. The geometric interpretation. The Cauchy method. Characteristics and developable characteristics.

We shall consider a first order partial differential equation in two variables x,y; the unknown function is z, p and q are partial derivatives of z with respect to x and y. Let

$$F(x,y,z,p,q) = 0 \tag{39}$$

be this equation. *We shall assume that the given function* F *admits continuous partial derivatives up to order two, throughout the domain in which these variables are taken. We look for solutions* $z = \phi(x,y)$ *possessing not only first partial derivatives* p *and* q *but also second continuous derivatives.* We shall regard these solutions as defining surfaces in three-dimensional space (x,y,z); these surfaces will be the *integral surfaces.* If a point $M(x,y,z)$ is given and if through this point there passes an integral surfaces S, then the coefficients p,q of the tangent plane at the point M of this surface satisfies equation (89). The normals at M to the integral surfaces are situated on a cone defined by equation (39); the tangent planes at M to these surfaces, have as their envelope the supplementary cone which we will refer to as *the elementary cone* and we denote this by $C(M)$. The equation of this cone $C(M)$ is obtained by determining the envelope of the plane

$$\zeta - z = p(\xi - x) + q(\eta - y) \quad , \tag{40}$$

where ξ, η, ζ are free coordinates and p and q are related by equation (39). Letting P,Q denote the partial derivatives of the function F with respect to p,q respectively, then we eliminate p and q from equations (39), (40) and the equation

$$\frac{\xi - x}{P} = \frac{\eta - y}{Q} \tag{41}$$

(no. 59). The problem of integration of equation (39) can be stated in the following geometric form: *To determine surfaces* S *tangent at each of their points* M *to the elementary cone* $C(M)$ *corresponding to this point.*

Let S be such a surface and $z = \phi(x,y)$ its equation. At each point M of S, the cone $C(M)$ is tangent to S. At this point, the functions P,Q have well defined values. *Firstly, we shall assume that* $P(x,y,z,p,q)$ *and* $Q(x,y,z,p,q)$ *are not simultaneously zero on the region of* S *in question.* The equality (41) defines at each point M of this region, a direction in the tangent plane; this is the direction of the generator of contact of the cone $C(M)$ and of S. There exists on S a family of curves Γ which are tangent, at each of their points M, to the cone $C(M)$; their projections on the plane Oxy are solutions of the differential equation

$$\frac{dx}{P} = \frac{dy}{Q} \tag{42}$$

where we assume that, in P and Q, z,p,q are replaced by $\phi(x,y)$, $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ respectively. The existence of these curves is guaranteed by the above hypothesis; through each point of S, there passes one and only one of them. Since on S we have $dz = p\,dx + q\,dy$, then we can form a new ratio equal to the ratios (42) along the curves Γ: on these curves, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{P_p + Q_q} \quad . \tag{43}$$

But, on S, we also have, where the second derivatives of z are denoted by r,s,t (a notation due to Monge)

$$dp = r\,dx + s\,dy \ , \qquad dq = s\,dx + t\,dy \ ,$$

such that on the curves Γ, the ratios (43) are equal to

$$\frac{dp}{P_r + Q_s} = \frac{dq}{P_s + Qt} \quad . \tag{44}$$

The above ratios can be written in another way. On S, equation (39) is satisfied. On differentiating the first member of (39) with respect to x and y, where x,p,q are replaced by $\phi(x,y)$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$ respectively, we obtain

$$X + pZ + \quad + Qs = 0 \ , \qquad Y + qZ + Ps + Qt = 0 \quad , \tag{45}$$

by setting

$$X = \frac{\partial F}{\partial x} \ , \qquad Y = \frac{\partial F}{\partial y} \ , \qquad Z = \frac{\partial F}{\partial z}$$

$$\left(P = \frac{\partial F}{\partial p} \ , \quad Q = \frac{\partial F}{\partial q}\right) \quad .$$

On taking account of the equalities in (45), the ratios (43) and (44) show that along the curves Γ, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{P_p+Q_q} = \frac{-dp}{X+pZ} = \frac{-dq}{Y+qZ} \quad . \tag{46}$$

In these equalities z,p,q are replaced by $\phi(x,y)$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$ respectively. *But when we assume that* x,y,z,p,q *are independent variables, then the differential system* (46) *admits solutions and defines* y,z,p,q, *for example, as functions of* x. If we take as initial values the coordinates x_0, y_0, z_0 of a point M_0 of S and the coefficients p_0, q_0 of the tangent plane to S at this point, then on account of the hypothesis $P(x_0,y_0,z_0,p_0,q_0) \neq 0$. (If this number happens to be zero, then $Q(x_0,y_0,z_0,p_0,q_0)$ will be different from zero and the roles of x and y can be permuted). The function P is nonzero in the neighborhood of the values x_0, y_0, z_0, p_0, q_0 and since P,Q,Z,X,Y have continuous first partial derivatives, there exists a single solution corresponding to these conditions. This solution defines in the space x,y,z a curve which coincides with the curve Γ passing through M_0 and a series of planes passing through the points M of this curve with coefficients p, q-1. These planes are, necessarily, the tangent planes to S along Γ, since the solution is unique. The curves Γ and the planes tangent to S along these curves are thus obtained independently from the nature of the surface S: it suffices to know a point M_0 on each curve Γ and the tangent plane Π_0 at this point. Thus we have:
If there exists integral surfaces of the type in question, then such surfaces will be engendered by the curves Γ *and will be tangent along these curves to the planes; these curves and planes can be obtained directly by integration of the system* (46).

CHARACTERISTICS AND DEVELOPABLE CHARACTERISTICS

The system (46) admits F as a first integral. We have, in fact, the integrable combination

$$Xdx + Ydy + Zdz + Pdp + Qdq \quad .$$

We shall confine our attention to solutions of the system for which $F = 0$. They will be provided by this equation and by three other first integrals G, H, K which along with F constitute a fundamental system. The solutions will be given by

$$F(x,y,z,p,q) = 0 \ , \qquad G(x,y,z,p,q) = a$$

$$H(x,y,z,p,q) = b \ , \qquad K(x,y,z,p,q) = c \ ,$$

where a,b,c are arbitrary constants (in a certain domain). Effectively, the system satisfies the conditions of the existence theorem, where we still assume that one of the denominators is nonzero in the domain in question.

More precisely, we assume that P *and* Q *are both not simultaneously zero at each point of the domain in question.* We can take x or y as the independent variable; the solutions can be written in the form

$$y = f(x,x_0,y_0,z_0,p_0,q_0) \ , \qquad z = g(x,x_0,y_0,z_0,p_0,q_0) \ ,$$

$$p = h(x,x_0,y_0,z_0,p_0,q_0) \ , \qquad q = k(x,x_0,y_0,z_0,p_0,q_0) \ ,$$

where x_0, y_0, z_0, p_0, q_0 are the initial conditions. The first two equations define a space curve in x,y,z. The curves thus defined depend on three parameters [x_0 can be fixed and we have $F(x_0,y_0,z_0,p_0,q_0) = 0$] and in fact form a complex. We call these curves the *characteristics* of equation (39). The curves Γ as defined above are characteristics. The other two equations determine, at each point $M(x,y,z)$, of a characteristic Γ, the numbers p and q such that $dz = pdx + qdy$; these are the coefficients of a tangent plane to Γ at M. These tangent planes along Γ have as their envelope, a developable surface known as the *developable characteristic*. The system of (46) is *the differential system of the characteristics*; this defines the characteristics and the developable characteristics. We also say that the characteristic Γ is *the support* of the developable characteristic, namely, the envelope of the tangent planes which are associated to it. With these definitions stated, the above result can be expressed as follows:
An integral surface S *on which* P *and* Q *are not simultaneously zero is a locus of characteristic curves and is the envelope of the developable characteristics whose supports are these curves.* Moreover, it can be seen how these must be associated to the characteristic curves which are susceptible to engender an integral surface S of the type in question. Let Λ be a curve possessing a continuous tangent, taken on S and intersecting the characteristics Γ. If M_0 is a point of Λ, with coordinates (x_0,y_0,z_0), then the characteristic Γ passing through M_0 will correspond to the initial conditions x_0, y_0, z_0, p_0, q_0, and q_0 must satisfy the condition

$$dz_0 = p_0dx_0 + q_0dx_0 \ , \qquad (F(x_0,y_0,z_0,p_0,q_0) = 0) \ ,$$

whose geometric interpretation is the following: the tangent plane of the developable characteristic with support Γ must contain the tangent to Λ at the point M_0.

SINGULAR POINTS AND SINGULAR INTEGRALS

The points of the x,y,z space for which $F(x,y,z,p,q) = 0$, $P(x,y,z,p,q) = 0$, $Q(x,y,z,p,q) = 0$, may in fact be isolated points, formed by lines on surfaces. In the first two cases, can only be zero at the points or on the lines described on the integral surfaces, which may be considered by suppressing these points or lines; such surfaces will in fact be engendered by characteristics.

Let us assume that the three equations $F=0$, $P=0$, $Q=0$ are satisfied on a surface Σ obtained by eliminating p and q from these equations. When Σ is an integral surface, then the above considerations no longer apply; Σ will, at the very least *be a singular integral.* In order for this to be the case, it is necessary and sufficient that the equations $F(x,y,z,p,q) = 0$, $P = 0$, $Q = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ should be compatible. Now, on differentiating the first with respect to x and y and taking account of the second and third, we see that we obtain

$$X + pZ = 0 , \qquad Y + qZ = 0 .$$

These conditions will be sufficient for Σ to be an integral surface when we do not simultaneously have $X=Y=Z=0$. We thus see, *that in general, there will be a singular integral if there exists a surface* Σ *on which* $F=0$, $P=0$, $Q=0$, $X+pZ=0$, $Y+qZ=0$.

273. The existence of integral surfaces. The Cauchy problem. Darboux's method. The analytic case.

On restricting our attention to the real elements, we see that an equation in partial derivatives (as is the case with a differential equation), cannot have any solution, even if it is algebraic. For example, the equation

$$(p^2+q^2)(x^2+y^2) + 1 = 0$$

cannot be satisfied by any collection of real numbers x,y,p,q, but it is sufficient to change z to iz, keeping x and y the same, in order to obtain solutions. We remark that, if the equation is not analytic, it cannot have any solution, even in the complex domain. For example, if $w(u)$ is a nonanalytic function having derivatives up to order 3, a function which cannot be defined for u complex (we could take $w(u)$ to be a function whose fourth derivative is a function without derivative), then the equation

$$p^2 + \omega^2(z) + \omega^2(y) + \omega^2(x) + 1 = 0$$

has neither a real solution or an analytic solution, since x,y,z are necessarily real, and hence p also.

We will need to hypothesize that equation $F=0$ is satisfied for values of p and q when the point x,y,z belongs to a certain domain, in real space if x,y,z are taken to be real, or for complex space when there is no restriction to real values, but, in this second case, when F is assumed to be analytic.

We shall firstly consider the reals. We assume that, in a domain Δ *of ordinary space* x,y,z, *the equation* $F=0$ *is satisfied for suitable values of* p *and* q; *have, the elementary cones* $C(M)$ *exist.* We shall always assume that the drivatives of F exist to order 2 and are continuous.

THE EXISTENCE OF INTEGRAL SURFACES

As a consequence of the analysis in no. 272, we will prove the existence of integral surfaces by defining an integral surface S passing through a given curve Λ satisfying certain conditions. *Thus we are led to a solution of the Cauchy problem.* We shall make the following hypotheses: Λ *is an arc of a simple continuous curve, having, at each of its points* M_0, *a tangent which varies continuously, which is not a generator of the cone* $C(M_0)$, *and such that for this tangent* M_0T_0 *we can take a tangent plane to the cone* $C(M_0)$. Putting it another way, along Λ, the coordinates x_0, y_0, z_0 of M_0 are functions of a parameter ν having continuous derivatives with respect to ν, and there exists functions p_0 and q_0 of ν such that

$$F(x_0,y_0,z_0,p_0,q_0) = 0 \ , \qquad z_0' = p_0x_0' + q_0,y_0' \quad , \tag{47}$$

for which $x_0'Q_0 - y_0'P_0$ is nonzero. *Moreover, we assume that* x_0'', y_0'', z_0'' *exist and are continuous.*

There exist such arcs Λ. Since given a point M_0 with corresponding elementary cone $C(M_0)$, and with Π_0 one of its tangent planes, then it suffices to take Λ to be an arc of a curve whose tangent M_0T_0 is situated in the plane Π_0, but is not identified with the generator of contact of Π_0. Given that the system in (47) is satisfied at the elected point M_0, we can solve with respect to p_0 and q_0 in the neighborhood of this point since the functional determinant is nonzero.

DARBOUX'S METHOD

Let us consider the differential system of characteristics (46) and its general integral satisfying the initial conditions x_0, y_0, z_0, p_0, q_0 where these numbers are functions of ν along Λ. We can state this general condition in terms of a parameter u by denoting the common value of the ratios (46) by du and taking 0 as the initial value of u. The solution will be of the form

$$\begin{cases} x = \xi(u,x_0,y_0,z_0,p_0,q_0) \ , & y = \eta(u,x_0,y_0,z_0,p_0,q_0) \ , \\ z = \zeta(u,x_0,z_0,p_0,q_0) \ , & p = \lambda(u,x_0,y_0,z_0,p_0,q_0) \ , \\ q = \mu(u,x_0,y_0,z_0,p_0,q_0) \ . & \end{cases} \tag{48}$$

The first three equations define a surface S in terms of the parameters u and v. This will be an integral surface when the values of p and q are the coefficients of the tangent plane, i.e. when the Pfaffian equation

$$dz = pdx + dy \quad ,$$

is satisfied. Now, we already have

$$\frac{\partial z}{\partial u} = p\frac{\partial x}{\partial u} + q\frac{\partial y}{\partial u} \quad , \tag{49}$$

since the ratios (46) are equal at du. It is necessary to establish that we also have

$$H(u,v) \equiv \frac{\partial z}{\partial v} - p\frac{\partial x}{\partial v} - q\frac{\partial y}{\partial v} = 0 \quad .$$

On account of the hypotheses made on the derivatives of F, the functions ξ, η, ζ admit derivatives with respect to x_0, y_0, z_0, p_0, q_0; on solving (47), p_0, q_0 admit derivatives with respect to v. The function $H(u,v)$ exists. Moreover, on referring back to no. 152, we see that

$$\frac{\partial^2 z}{\partial v\partial u} , \qquad \frac{\partial^2 x}{\partial v\partial u} , \qquad \frac{\partial^2 y}{\partial v\partial u}$$

exist and we can invert the order of differentiation. We have

$$\frac{\partial H}{\partial u} = \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial p\partial x}{\partial u\partial v} - \frac{\partial q\partial y}{\partial u\partial v} - p\frac{\partial^2 x}{\partial u\partial v} - q\frac{\partial^2 y}{\partial u\partial v} \quad .$$

On the other hand on differentiating the identity (49) with respect to v, we have

$$\frac{\partial^2 z}{\partial u\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial q}{\partial v}\frac{\partial y}{\partial u} - p\frac{\partial^2 x}{\partial u\partial v} - q\frac{\partial^2 y}{\partial u\partial v} = 0 \quad .$$

Comparing these equalities and those in (46) yields

$$\begin{aligned}\frac{\partial H}{\partial u} &= \frac{\partial p}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial q}{\partial v}\cdot\frac{\partial y}{\partial u} - \frac{\partial p}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial q}{\partial u}\cdot\frac{\partial y}{\partial v} \\ &= P\frac{\partial p}{\partial v} + Q\frac{\partial q}{\partial v} + X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}\right) .\end{aligned}$$

But if we also have $F \equiv 0$, then F is a first integral and $F(x_0,y_0,z_0,p_0,q_0) = 0$; it then follows that

$$P\frac{\partial p}{\partial v} + Q\frac{\partial q}{\partial v} + X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\frac{\partial z}{\partial v} \equiv 0 \quad ,$$

such that we have

$$\frac{\partial H}{\partial u} = Z\left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v} - \frac{\partial z}{\partial v}\right) = -ZH \quad . \tag{50}$$

On account of the hypothesis (47), $H(0,v)$ is zero. It thus follows that $H(u,v) \equiv 0$. This can be seen by reapplying the method adopted in no. 77. If for some value v, $H(u,v)$ is not always zero in the interval of the values of u for which the function is defined, then $H(u,v)$ is nonzero for any point in the interval u', u_1' and will be zero for $u = u'$ for example,

since $H(0,v) = 0$. If u'' is taken to be between u' and u_1', then we have, on integrating equation (50),

$$H(u,v) = H(u'',v)e^{\alpha}, \qquad \alpha = -\int_{u''}^{u} Z\,du \quad ,$$

with $u' < u < u''$. When u tends towards u', $H(u,v)$ tends toward $H(u',v)$ which is nonzero on account of this equality, contrary to hypothesis. Thus, $H(u,v)$ is zero, the surface S is an integral surface. Moreover, p and q possess derivatives with respect to u and v which are continuous; it follows that r,s,t exist and are continuous. The surface S possesses all the properties, *a priori*, induced by the solutions. Consequently, we arrive at the following

Theorem. *If* F *has continuous partial derivatives with respect to* x,y,z,p,q *up to order* 2 *and if* $F = 0$ *is true for values of* p *and* q *in a domain* Δ, *then there exist integral surfaces. The nonsingular integral surfaces are obtained by the solution of the Cauchy problem for curves* Λ *possessing the properties as indicated above.*

When we are given a curve Λ whose tangents to the points M_0 are not generators of the elementary cones $C(M_0)$, then two cases are possible: either there does not exist tangent planes to Λ tangent to the cones $C(M_0)$, in which case the Cauchy problem is impossible; or there do exist such planes, in which case there are several of them in general. When one of these planes is chosen at a particular point M_0, then the others will be determined by continuity. The integral surface will then, in turn, be determined. It will be engendered by the characteristic curves passing through the points M_0 of Λ and tangent the points M_0 to the tangent planes common to Λ and $C(M_0)$ thus determined. This surface is unique.

PROPERTIES OF INTEGRAL SURFACES

If two integral surfaces are tangent at a point $M_0(x_0,y_0,z_0)$ and if P_0 and Q_0 are not simultaneously zero at this point, i.e. the point is non-singular, then these surfaces have the characteristic curve Γ in common, which corresponds to the initial conditions x_0, y_0, z_0, p_0, q_0 and are tangent to each other along this curve since the characteristic developable having Γ as its support is determined uniquely by these initial conditions, as is Γ.

The integral surfaces depend on an arbitrary function. Let us assume, for example, that a plane $x = x_0$ is not tangent to the cones $C(M_0)$ having their vertices in this plane and the tangent planes to one of these cones intersect this plane along some lines belonging to the angles of vertex M_0. We may then define curves Λ with the above mentioned properties by taking

$$x = x_0 , \qquad z_0 = \psi(y_0) , \tag{51}$$

where $\psi(y_0)$ admits continuous derivatives of the first two orders. Moreover, $\psi(y_0)$ is constrained to satisfy the inequalities necessary to guarantee the existence of the tangent planes common to Λ and $C(M_0)$. The quantities p_0 and q_0 will depend on $\psi(y_0)$ and $\psi'(y_0)$ since they are determined by conditions (47) which, here, take the form

$$F(x_0,y_0,\psi(y_0),p_0,q_0) = 0 , \qquad \psi'(y_0) = q_0 .$$

On substituting into the formulae (48), the values of x_0, y_0, z_0, p_0, q_0 thus defined by the arbitrary function ψ, we determine the integral surfaces in terms of ψ and ψ'.

THE ANALYTIC CASE

If the function F is analytic, then the differential system of the characteristics is analytic and the solutions given by the first integrals on those of the form (48), are analytic. In particular, we see that in the form (48), x,y,z,p,q are analytic functions of u with initial conditions x_0, y_0, z_0, p_0, q_0. Regarding the Cauchy problem, if we take Λ to be an analytic curve, x_0, y_0, z_0 to be analytic functions of v, then the corresponding integral surface will have coordinates which will be analytic functions of u and v. This will be an analytic surface.

If Λ is not analytic, then the corresponding surface S will not, in general, be analytic. In particular, if we should take Λ to be defined as above, by equations (51), then it suffices to the function $\psi(y_0)$ to be nonanalytic in order for S to be nonanalytic, since the section of an analytic surface through a plane, is an analytic curve. Consequently:
The analytic integral surfaces of an analytic partial differential equation only constitute part of the set of integral surfaces. The most general integral is nonanalytic.

274. Integral curves. The Monge equation.

We have seen in no. 272 how the equation of the cone $C(M)$ is obtained. We assume that this is actually a cone, i.e. the equation is nonlinear. Take

$$\Phi(\xi-x,\ \eta-y,\ \zeta-z,\ x,\ y,\ z) = 0 \tag{52}$$

to be the equation of this cone. Curves of the (x,y,z) space which, at each of their points M, are tangent to one of the generators of the corresponding elementary cone $C(M)$, are determined by the equations

$$\frac{dx}{\xi-x} = \frac{dy}{\eta-y} = \frac{dz}{\zeta-z} ,$$

where the denominators are related by equation (52); they are homogeneous with respect to these denominators. The equation

$$\Phi(dx,dy,dz,x,y,z) = 0 \tag{53}$$

which, at each point of M, defines the directions of the tangents to the curves in question, is known as *the Monge equation associated to equation* (39). The solution curves of this equation (53) depend on an arbitrary function. For, we can always express their projections onto the Oxy plane arbitrarily; for example, in the form $y = f(x)$, where $f(x)$ is differentiable and z is then given as a function of x by the differential equation

$$\Phi(dx,f'(x)dx,dz,x,f(x),z) = 0$$

whose solution again depends on an arbitrary constant. It follows that, through a point $M_0(x_0,y_0,z_0)$ there passes an infinity of these curves, depending on one parameter, whilst the characteristic curves passing through this point only depend on an arbitrary parameter p_0 (where q_0 is related to p_0 by the condition $F_0 = 0$). *These curve solutions of the associated Monge equation* $F = 0$, *which are considered to be particular examples of characteristic curves, are called the integral curves of the equation* $F = 0$.

Remark. For a linear equation, the elementary cone reduces to a line and the integral curves are uniquely the characteristics.

THE CAUCHY PROBLEM

In order to solve the Cauchy problem, we assume that the curve Λ is not an integral curve. Given a curve Λ which is an integral curve without being a characteristic, then the curve may be seen to be an envelope of characteristic curves. Through each point M_0 of Λ, there passes a characteristic curve Γ tangent to Λ for which the comment tangent is a generator of the cone $C(M_0)$; let π_0 be the tangent plane to $C(M_0)$ along this generator, and p_0, q_0 the coefficients of this tangent plane. The curve Γ corresponds to the initial conditions (x_0,y_0,z_0,p_0,q_0), but if x_0', y_0', z_0' are taken to be the direction parameters of the tangent to Λ at the point M_0, then the system of equations (47) at p_0 and q_0 is now no longer solvable, since we have $x_0'Q_0 - y_0'P_0 = 0$. However, p_0 and q_0 will still be well defined functions of the parameter v fixing the position of M_0 on Λ and these functions will have derivatives. Darboux's method again applies here, for the surface S engendered by the curves Γ will be an integral surface. But Λ will be a singular line of this surface; u and v cannot normally be expressed as functions of x and y at the points M_0.

275. The method of Lagrange and Charpit. Complete integrals.

Let us consider a family of surfaces depending on two parameters a and b, given explicitly by

$$V(x,y,z,a,b) = 0 \ . \tag{54}$$

We assume here that the function V is analytic and $\frac{\partial V}{\partial z} \neq 0$ in general. The coefficients p and q of the tangent plane to a surface belonging to the family are given by

$$\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z} = 0 \ , \qquad \frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial x} = 0 \ . \tag{55}$$

If we eliminate a and b from equations (54) and (55) by assuming that two of the equations are solvable at a and b, and on substituting the values obtained into the third, we obtain an equation

$$F(x,y,z,p,q) = 0 \tag{56}$$

whose first member is analytic in $x,y,z,p,q,$ and which is satisfied for any x,y,a,b when z is replaced by its value taken from (54) and p and q by its derivatives. This is, in fact, the partial differential equation of the surfaces (54).

Remark. More generally, equation (54) can always be seen to be solved with respect to a, for example, since $\frac{\partial V}{\partial a}$ is generally nonzero. By substituting for a the value deduced from (55), we obtain two equations in which b is no longer contained. At least one is solvable in b, and on bringing the value obtained for b into the other, we recover equation (56). For, if the derivative with respect to b is zero in the two equations thus deduced from (55), then these equations would yield, for p and q, values which depend neither on a nor on b. The surfaces (54) will be integral surfaces of a total differential equation; they are integral surfaces which can only depend on a single arbitrary constant. *The surfaces (54) do not depend on two parameters but on a single parameter alone.*

OTHER SURFACES SATISFYING EQUATION (54)

If we assume that the parameter b in (54) is replaced by a differentiable function of a, $b = \lambda(a)$, and if the one-parameter envelope of the surfaces thus defined, is considered, then this surface envelope is also an integral surface of equation (56). Effectively, through every ordinary point of this surface envelope (a surface whose existence is assumed) there passes a surface (54) which is tangent to it; equation (56) is satisfied for values of x,y,z,p,q at this point of the surface (54) and consequently is satisfied by the surface envelope.

Similarly, if the 2-parameter surfaces (54) have an envelope, then this surface envelope is also a solution of equation (56).

THE LAGRANGE METHOD. THE COMPLETE INTEGRAL. THE GENERAL INTEGRAL. THE SINGULAR INTEGRAL.

Let us assume, given *a priori* a partial differential equation which is analytic, that a family of 2-parameter surfaces (54) is known and which satisfies (56) for any of these parameters. Such a family of surface solutions was named *a complete integral* by Lagrange. On account of what we have just said, these 2-parameter surfaces have a unique partial differential equation which is necessarily equation (56). Moreover, *we obtain other integral surfaces by proceeding as we did above: we set* $b = \lambda(a)$ *and we consider the envelope of the surfaces thus obtained; we consider also the envelope, if it exists, of the 2-parameter surfaces defined by (54). These surface envelopes are in fact the integral surfaces.*

Lagrange showed that this provided all solutions. Let $z = \phi(x,y)$ be a solution of equation (56), $p = \frac{\partial\phi}{\partial x}$, $q = \frac{\partial\phi}{\partial y}$. Let $z = \phi(x,y)$ be a solution of equation (56), $p = \frac{\partial\phi}{\partial x}$ and $q = \frac{\partial\phi}{\partial y}$. *For each value of* x,y, the five numbers x,y,z,p,q satisfy equation (56). Now this equation is the result of eliminating a and b from equations (54) and (55). To the numbers x,y,z,p,q there also corresponds a system of values a,b obtained by solving equations (54) and (55), and such that these two equations are satisfied for the system of values x,y,z,p,q,a,b. *These numbers* a *and* b *are functions of* x *and* y. To the function $z = \phi(x,y)$ there also corresponds two functions $a = A(x,y)$, $b = B(x,y)$ such that we have identically

$$V(x,y,\phi,A,B) = 0\ , \qquad \frac{\partial V}{\partial x} + \frac{\partial V}{\partial z}\frac{\partial\phi}{\partial x} = 0\ ,$$

$$\frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}\cdot\frac{\partial\phi}{\partial y} = 0\ . \tag{57}$$

These functions A and B can be obtained by the usual means of solving two of these equations (the functional determinant being nonzero); they possess derivatives. On differentiating the first equation (57) with respect to x and y respectively and taking the other two into account, we obtain

$$\frac{\partial V}{\partial a}\frac{\partial A}{\partial x} + \frac{\partial V}{\partial b}\frac{\partial B}{\partial x} = 0\ ; \qquad \frac{\partial V}{\partial a}\frac{\partial A}{\partial y} + \frac{\partial V}{\partial b}\cdot\frac{\partial B}{\partial y} = 0\ . \tag{58}$$

As $\frac{\partial V}{\partial a}$ and $\frac{\partial V}{\partial b}$ are, in general, nonzero, the system in (58) can only be satisfied when we have

$$\frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} = 0\ . \tag{59}$$

This condition can be explained in two ways: B is a function of A (or A could be a function of B) and this function is differentiable; alternatively, A and B could be constants. In the second case, the surface in question $z = \phi(x,y)$, is a complete integral surface. In the first case, the surface $z = \phi(x,y)$ is defined by the conditions

$$V(x,y,z,A,\lambda(A)) = 0 , \qquad \frac{\partial V}{\partial A} + \frac{\partial V}{\partial B}\frac{d\lambda}{dA} = 0 ,$$

since equations (58) then reduce to a single equation. This is the 1-parameter envelope of surfaces obtained by replacing b by $\lambda(a)$ in the complete integral.

It remains to examine the case where we have

$$\frac{\partial V}{\partial a}(x,y,\phi,A,B) \equiv 0 , \qquad \frac{\partial V}{\partial b}(x,y,\phi,A,B) \equiv 0 .$$

This is the case where the complete integrals admit a 2-parameter envelope. We thus obtain the result (of Lagrange):

When a complete analytic integral of the analytic equation $F(x,y,z,p,q) = 0$ *is known,* $V(x,y,z,a,b) = 0$ *say, then we obtain the most general solutions of the equation by taking* b *to be some differentiable function* $\lambda(a)$ *of* a *and then eliminating* a *from the equations*

$$V(x,y,z,a,\lambda(a)) = 0 , \qquad \frac{\partial V}{\partial x} + \frac{\partial V}{\partial \lambda}\lambda'(a) = 0 .$$

In certain cases, we may have another integral when the complete integral surfaces admit a 2-parameter envelope. This surface envelope defines the singular integral.

The geometric interpretation of the general integral was stated above. We see that this integral depends on an arbitrary function $\lambda(a)$.

<u>On determining a complete integral (Lagrange and Charpit).</u>

In order to obtain a complete integral in every case, we supplement equation (56) by another equation

$$G(x,y,z,p,q) = a ,$$

where a is an arbitrary constant, such that the system $F = 0$, $G = a$ is completely integrable. The general integral of this system will depend on a and on the constant of integration b; this will be an integral of $F = 0$ depending on two parameters. When G is analytic, this complete integral will likewise be analytic.

The condition for complete integrability of the system

$$F(x,y,z,p,q) = 0 , \qquad G(x,y,z,p,q) = a$$

is $[F,G] = 0$ (no. 271), i.e. by using the notation X, Y, Z, P, Q to denote partial derivatives of F with respect to x,y,z,p,q

$$\frac{\partial G}{\partial x}P + \frac{\partial G}{\partial y}Q + \frac{\partial G}{\partial z}(P_p + Q_q) - \frac{\partial G}{\partial p}(X + pZ) - \frac{\partial G}{\partial q}(Y + qZ) = 0 .$$

It suffices that the function G should be a first integral of this linear homogeneous partial differential equation. It suffices to take $G(x,y,z,p,q)$ to be a first integral of the differential system (identical to those of the

characteristics)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{P_p + Q_q} = -\frac{dp}{X+pZ} = -\frac{dq}{Y+qZ}, \tag{60}$$

a first integral which is analytic. Consequently:
In order to obtain a complete integral, we look for a first integral of the differential system (60), $G(x,y,z,p,q)$ *say, such that it is possible to solve the system* $F=0$, $G=a$ *with respect to* p *and* q. *We proceed to integrate the total differential equation* $dz = pdx + qdy$ *thus defined, which yields a complete integral.*

Remark. When x is regarded as the independent variable in the system in (60) ($P \neq 0$), than we know that there exists a fundamental system of first integrals, a system in which F can be one of the integrals. The functional determinant of the functions of this system, taken with respect to y,z,p,q, is nonzero. It than follows that the functional determinant of F and at least one of the other first integrals, taken with respect to p and q, is nonzero.

276. A comparison of the two methods.

The hypotheses that we imposed on F are not the same in the two methods described, but in dealing with the Lagrange method, we had allowed ourselves the assumption (of Cauchy's method) that F possessed a certain number of partial derivatives in x,y,z,p,q. We shall compare the two methods without involving such different hypotheses by assuming, for example that F is analytic.

In fact the Lagrange method is often advantageous. It may happen that a complete integral was obtained by direct methods, without having to integrate the system of equations in (60). Even if this is not the case, the system in (60), which is the system of characteristics (46) of the Cauchy method was not completely integrated. It suffices to obtain a single first integral in order to reduce matters to integration of a total differential equation, i.e. an ordinary differential equation when Mayer's method is applied (no. 268).

Remark. When a first integral (distinct from F) of the system of characteristics is obtained, then the integration of the system reduces, for example, to the integration of the system formed by the first three ratios (60) or (46), in which p and q are replaced as a function of x,y and a constant a, in terms of the first integrals $F=0$, $G=a$. But this system involving two equations is not so simple as the total differential equation $dz = pdx + qdy$.

CHARACTERISTICS DEDUCED FROM THE COMPLETE INTEGRAL

A general integral surface S is the envelope of a family of surfaces $V(x,y,z,a,\lambda(a)) = 0$. At an ordinary point M_0, the surface S and the enveloped surface are tangent and they remain so along a line Γ which is the characteristic from the theory of envelopes. But following the theory of Cauchy (no. 273), these two integral surfaces are tangent along the characteristic curve eminating from M_0 and tangent at M_0 to the tangent plane of S. These characteristic curves defined in those two different ways are unique; it therefore follows that they coincide. Thus:
The characteristic curves arising in the Cauchy method are the contact curves of the general integrals with the complete integral surfaces which admit these as envelopes. The equations of these curves are

$$V(x,y,z,a,\lambda(a)) = 0 , \qquad \frac{\partial V}{\partial a} + \frac{\partial V}{\partial b}\frac{d\lambda}{da} = 0 , \tag{61}$$

where $\lambda(a)$ is an arbitrary differentiable function for fixed a; $\lambda(a)$ and $\frac{d\lambda}{da}$ are therefore arbitrary and we obtain the following result:

The complex of characteristic curves is defined by

$$V(x,y,z,a,b) = 0 , \qquad \frac{\partial V}{\partial a}(x,y,z,a,b) + \frac{\partial V}{\partial b}(x,y,z,a,b)c = C$$

where a,b,c *are arbitrary parameters.* The characteristic developables are then obtained by supplementing these equation by

$$\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z} = 0 , \qquad \frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial z} = 0 ,$$

which yield p and q.

INTEGRAL CURVES

The integral curves in no. 274 are the envelopes of characteristic curves. They are obtained by supplementing equations (61) by the equation obtained by differentiation with respect to a of the second of these equations. They depend on the arbitrary function $\lambda(a)$.

The Cauchy problem. The determination of integral surfaces passing through a given curve may be obtained directly from the complete integral. A general integral surface is given by the equations (61) from which a can be eliminated. We need to find $\lambda(a)$ in order for this surface to contain the given curve Λ defined by $x = f(t)$, $y = g(t)$, $z = h(t)$. The function $\lambda(a)$ must satisfy the conditions

$$V(f,g,h,a,\lambda(a)) = 0 ,$$

$$\frac{\partial V}{\partial a}(f,g,h,z,\lambda) + \frac{\partial V}{\partial b}(f,g,h,a,\lambda)\,\frac{\partial \lambda}{\partial a} = 0 \quad . \tag{62}$$

The first expresses a as a function of t, and the second condition must be satisfied by this function of t. Now, on differentiation of the first condition, where a is regarded as being replaced by its value as a function of t, and on taking account of the second, we have remaining

$$\frac{\partial V}{\partial x} f' + \frac{\partial V}{\partial y} g' + \frac{\partial V}{\partial z} h' = 0 \quad . \tag{63}$$

The function a of t must satisfy the first relation (62) and condition (63). This determines $\lambda(a)$ when the system thus obtained is solvable at a and $\lambda(a)$. Putting it another way, the parameters a and b appearing in the complete integral will be defined as a function of t by the conditions

$$V(f,g,h,a,b) = 0 \ , \qquad \frac{\partial V}{\partial x}(f,g,h,a,b)f' + \frac{\partial V}{\partial y}(\ldots)g' + \frac{\partial V}{\partial z}(\ldots)h' = 0. \tag{64}$$

This expresses the fact that the surface corresponding to these values a and b is tangent to the curve Λ. Conversely, if we consider the surfaces of the complete integral which are tangent to Λ, then these surfaces depend on the parameter t. If they possess an envelope, then this surface envelope will contain Λ. In the case where Λ is a characteristic curve, all the integral surfaces passing through this curve respond to the question, the problem is indeterminate. The same situation arises when Λ lies on a singular integral. If Λ is not a characteristic curve, then equations (64) yield a and b as functions of t and, on taking the envelope of the surfaces obtained by replacing a and b by these functions of t in the complete integral, we obtain the surface in question, which can be decomposed. Thus we have:
If Λ is not a characteristic curve and not situated on the singular integral, then the Cauchy problem relative to Λ may be solved by taking the envelope of the surfaces of the complete integral which are tangent to Λ.

Remark. We will obtain the integral surface tangent to a given surface Σ by taking the envelope of the surfaces of the complete integral which are tangent to Σ. There will be indeterminacy when Σ is the singular integral.

Complete integrals deduced from the Cauchy method.

These are obtained by considering the integral surfaces depending on 2-parameters. For example, the Cauchy problem may be solved by taking a curve Λ depending on 2-parameters. We may also consider the integral surface engendered by the characteristic curves passing through a point $M_0(x_0,y_0,z_0)$ and assume that two of the coordinates of M_0 are parameters, with the third coordinate fixed. Such a surface has a conic point at M_0, where the tangent cone is the cone $C(M_0)$; the Darboux method shows that the characteristic curves tangent to the generators of this cone at M_0, engender an integral surface.

Remark. Linear equations featured prominently in the work of Lagrange and Cauchy. The characteristic curves form a congruence, whereas the set of curves and characteristic developables is in fact a complex.

277. Some particular cases. Examples.

I. CLAIRAULT EQUATION

Such equations are of the form

$$z = px + qy + f(p,q) \quad .$$

When p and q are replaced by arbitrary constants a and b, then we obtain the equation of a plane which defines a solution since under these conditions, we have, $p = a$, $q = b$. A complete integral is therefore $z = ax + by + f(a,b)$. The general integrals are obtained by taking $b = \lambda(a)$ and determining the envelope; these are developable surfaces. The 2-parameter envelope defined by

$$x = -f'_a(a,b) \;, \qquad y = -f'_b(a,b) \;, \qquad z = ax + by + f(a,b)$$

is the singular integral Σ. The characteristic curves are the rectilinear generators of the developable surfaces; these are the tangents of the singular integral Σ. If Γ is one of these tangents, then the characteristic developable of the support Γ is the tangent plane to Σ at the point of contact of Σ and Γ.

II. EQUATIONS WITH SEPARABLE VARIABLES

These are equations of the form

$$f(x,p) = g(y,q) \; .$$

The differential equations of the system of characteristics are

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial g}{\partial q}} = \frac{dz}{p\frac{\partial f}{\partial p} - q\frac{\partial g}{\partial q}} = \frac{-dp}{\frac{\partial f}{\partial x}} = \frac{dq}{\frac{\partial f}{\partial y}} \quad .$$

The first and fourth ratio give rise to the integral combination

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial p}\, dp = 0 \quad .$$

We therefore have

$$f(x,p) = a \;, \qquad g(y,q) = a \;,$$

where a is constant, it follows that p and q are expressed respectively as functions of x and y alone and the integration of the total differential equation $dz = pdz + qdy$ is achieved by means of two quadratures.

Example. Consider the equation $p^2x - qy^2 = 0$. We have $p^2x = a = qy^2$,

$$dz = \sqrt{\frac{a}{x}}\, dx + \frac{a}{y^2}\, dy ,$$

and the complete integral

$$z = 2\sqrt{ax} - \frac{a}{y} + b .$$

The characteristic curves are defined by this equation and by

$$\sqrt{\frac{x}{a}} - \frac{1}{y} + C = 0 .$$

Here, the cones $C(M)$ are of the second degree. The complete integral surfaces are algebraic and of the fourth degree; the integral surface which passes through an algebraic curve will be an algebraic surface, irreducible in general. There is no singular integral.

Remark. The equations of the form $F(x,p,q) = 0$ or $F(y,p,q) = 0$ enter into the class of equations with separable variables. $q = a$ for the first equation, $p = a$ for the first integrals of the differential system of characteristics.

III. EQUATIONS NOT CONTAINING z. Let $F(x,y,p,q) = 0$ be an equation not containing z. In order to obtain a complete integral, we can associate to it an equation of the form $G(x,y,p,q) = a$, not containing z. In order to express the fact that the system of these equations is completely integrable, we apply the condition $(F,G) = 0$ (no. 271); G will be a first integral of the simplified system

$$\frac{dx}{P} = \frac{dy}{Q} = -\frac{dp}{X} = -\frac{dq}{Y} . \tag{65}$$

Once G is known, p and q will be given by functions of x and y alone and we recover the integration of a total differential.

Example. Consider the equation $px^3 + q^2 + xy = 0$. The system (65) admits the first integral $q - 1/x$. We proceed to integrate the equation

$$dz = -\left[\frac{1}{x^3}\left(\frac{1}{x} + \frac{1}{a}\right)^2 + \frac{y}{x^2}\right] dx + \left(\frac{1}{x} + a\right) dy ,$$

and obtain the complete integral

$$z = \frac{1}{4x^4} + \frac{2a}{3x^3} + \frac{a^2}{2x^2} + \frac{y}{x} + ay + b ,$$

There is no singular integral.

IV. EQUATIONS NOT CONTAINING z AND ADMITTING $py - qx$ AS THE FIRST INTEGRAL OF THE CHARACTERISTIC SYSTEM. The system (65) must admit the integrable combination $pdy - qdx + ydp - xdq$, and it then follows that

$$Pq - Qp + yX - xY = 0 .$$

The function $F(x,y,p,q)$ is the integral of this partial differential equation; the associated differential system

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dp}{q} = -\frac{dq}{p}$$

admits the integral combinations $p^2 + q^2$, $x^2 + y^2$, and $py - qx$. The equation in question is of the form

$$F(p^2 + q^2,\ x^2 + y^2,\ py - qx) = 0 .$$

A complete integral of such an equation will be given by integrating the system

$$py - qx = a, \qquad p^2 + q^2 + \Phi(x^2 + y^2, a) , \tag{66}$$

where the second equation is deduced from (66) when $py - qx$ is replaced by a and solved for $p^2 + q^2$. We thus deduce the equation

$$(px + qy)^2 = (p^2 + q^2)(x^2 + y^2) - (py - qx)^2$$
$$= (x^2 + y^2)\Phi(x^2 + y^2, a) - a^2$$

and on denoting the last member of this equality by $\Psi^2(x^2 + y^2, a)$, we have

$$dz = pdx + qdy = \Psi(x^2 + y^2, a)\ \frac{xdx+ydy}{x^2+y^2} + a\ \frac{ydx-xdy}{x^2+y^2} ,$$

which yields the complete integral

$$z = a \text{ arc tg } \frac{x}{y} + \int \Psi(u,a)\, \frac{du}{2u} + b , \qquad u = x^2 + y^2 .$$

The surfaces defined by this equation are helicoids.

<u>Example</u>. The equation

$$(py - qx)^2 = c^2(1 + p^2 + q^2) ,$$

where c is a given constant, admits

$$z = a \text{ arc tg } \frac{x}{y} + \int \sqrt{u\, \frac{a^2 - c^2}{c^2} - a^2}\, \frac{du}{2u} + b$$

as a complete integral and one can proceed to integrate by setting the root equal to v.

Remark. In order for the system $F = 0$, $G = a$ to be completely integrable, it suffices to see that the condition $[F,G] = 0$ is a consequence of $F = 0$ (no. 271). We can account *a priori* for the condition $F = 0$ before stating the differential equations in (60). For example, if we have the equation

$$p + f(x,y,z,q) = 0 \ ,$$

then on replacing p by $-f$, the condition becomes

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial f}{\partial q} + \frac{\partial G}{\partial z}\left(q\,\frac{\partial f}{\partial q} - f\right) - \frac{\partial G}{\partial p}\left(\frac{\partial f}{\partial x} - f\,\frac{\partial f}{\partial z}\right) - \frac{\partial G}{\partial q}\left(\frac{\partial f}{\partial y} + q\,\frac{\partial f}{\partial z}\right) = 0 \ .$$

As p does not appear in the coefficients, we can assume G to be independent of p. $G(x,y,z,q)$ will be a first integral of the system

$$\frac{dx}{1} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{q\,\frac{\partial f}{\partial q} - f} = \frac{-dq}{\frac{\partial f}{\partial y} + q\,\frac{\partial f}{\partial z}} \ .$$

V. THE CASE WHERE A COMPLETE INTEGRAL CAN BE OBTAINED A PRIORI. Let us find surfaces S such that for a given point M of S, P the point of intersection of the normal at M with the plane Oxy, then the orthogonal projection of the segment P onto the plane Oxy is of given length a ($Oxyz$ rectangular). The partial differential equation is

$$z^2(p^2 + q^2) = a^2 \ . \tag{67}$$

The system of the equations of the characteristics is (to within a factor of 1/2)

$$\frac{dx}{pz^2} = \frac{dz}{qz^2} = \frac{dz}{(p^2+q^2)z^2} = \frac{-dp}{pz(p^2+q^2)} = \frac{-dq}{qz(p^2+q^2)} \ . \tag{68}$$

We have the integrable combination p/q .
Having taken account of the form of equation (67), we can take

$$\frac{p}{\cos\alpha} = \frac{q}{\sin\alpha} = \sqrt{p^2+q^2} = \frac{a}{z} \ ,$$

where α is a parameter, and we then proceed to integrate

$$dz = \frac{a\cos\alpha}{z}\,dx + \frac{a\sin\alpha}{z}\,dy \ ,$$

which yields the complete integral

$$z^2 = 2a(x\cos\alpha + y\sin\alpha) + \beta \ ,$$

where β is the second parameter. The complete integral surfaces are

parabolic cylinders of equal type whose line sections are parabolas situated in the planes parallel to Oz, having their axes in the plane Oxy and these are all equal. We can anticipate *a priori* this result in advance. Since on account of the property of the subnormal to a parabola, every parabola π with plane parallel to Oz with axis situated in the plane Oxy and with parameter a, is an integral curve (no. 274) and the cylinders admitting these parabolas π as line sections are integral surfaces. They depend on two parameters and hence give a complete integral. Two of these cylinders intersect along a parabola whose limit, when one of the cylinders tends towards the other, is a parabola of a line section. The characteristic curves are the parabolas π, the characteristic developable whose support is a parabola π, is the cylinder for which π is the line section. An integral surface can be obtained by taking an arbitrary curve Λ in the plane (Oxy) (Λ has a continuous tangent) and taking the locus of parabolas π whose vertex describes Λ and whose plane is normal to Λ.

The system in (68) can be completely integrated without too much trouble (we can cancel the factor z out of the denominator). We have $p = q\cot\alpha$ and from (67) $pz = a\cos\alpha$; the first two ratios give $y = x\,tg\,x + \lambda$, the first and the fourth are then written as

$$dx = -\frac{a\cos\alpha\, dp}{p(p^2+q^2)} = -\frac{a\cos^3\alpha\, dp}{p^3}$$

and give

$$x = \frac{a\cos^3\alpha}{2p^2} + \mu \quad .$$

We obtain the system of curves and characteristic developables in the form

$$y\cos\alpha - x\sin\alpha = \lambda', \qquad z^2\cos\alpha = 2ax + \mu' \quad ,$$

$$\frac{p}{\cos\alpha} = \frac{q}{\sin\alpha} = \frac{a}{z} \quad ,$$

where α, λ' and μ' are arbitrary constants. A more direct way of seeing this is when we consider surfaces S such that the part of the normal taken between the point of incidence and the plane Oxy has a constant length a. The spheres of radius a whose center is in Oxy are integral surfaces and define a complete integral. The general integral surfaces are canal surfaces; there exists a singular integral constituted by the two planes being the loci of points situated at the distance a from Oxy.

278. The generalized Lie solutions

With the intention of applying contact transformations to first order partial differential equations, Lie generalized the notion of a solution.

We say that the elements

$$x = f(u,v), \quad y = y(u,v), \quad z = h(u,v), \quad p = k(u,v), \quad q = \ell(u,v)$$

which depend on two parameters (u,v) and are differentiable, are *united* and *define a 2-dimensional multiplicity* if the condition

$$dz \equiv pdx + qdy$$

is satisfied. The set of points with coordinates x,y,z is the point-support of the multiplicity. This support can be reduced to a curve where p, $q-1$ are the coefficients of the tangent planes to this curve at the points of this curve: the multiplicity is the set of points of a curve and its tangent planes. The support may even reduce to a point; the multiplicity is the set of planes passing through this point. In general, the support is a surface and the multiplicity is the set of points of this surface and its tangent planes.

Following Lie's terminology we say that the generalized solution of the equation

$$F(x,y,z,p,q) = 0 \tag{69}$$

is every 2-dimensional multiplicity identically satisfying this equality. If p or q occur in this equation (which is now a partial differential equation), then the integral surfaces united with their tangent planes, are solutions. But there may well exist Lie solutions which are not defined by integral surfaces. For example, if equation (69) happens to be linear, then the multiplicity defined by a characteristic curve and its tangent planes constitute a solution in the Lie sense.

From Lie's point of view an equation of the type $F(x,y,z) = 0$ can be considered as a particular partial differential equation: it admits as a solution the multiplicity formed by the surface S defined by this equation, its tangent planes and its multiplicities defined by the points M of S and the planes passing through these points. A contact transformation (I, 134, 135) transforms a multiplicity of united elements into a multiplicity of united elements. If the first satisfies equation (69), the second satisfies the transformed equation obtained by replacing x,y,z,p,q in (69) by the values extracted from the inverse transformation. It may happen that the transformation of an integral surface of (69) is only a generalized solution of the transformed equation. Thus we have to consider the generalized solutions of the transformation if we wish to obtain, via the transformation, all of the solutions of the equation in question.

THE CASE OF TRANSFORMATIONS BY POLAR INVERSION

In particular, when we make a polar inversion to the united elements x,y,z,p,q there correspond the united elements X,Y,Z,P,Q defined by equations of the form

$$X = \phi_1(p,q,px+qy-z), \qquad Y = \phi_2(p,q,px+qy-z),$$

$$Z = \phi_3(p,q,px+qy-z), \qquad P = \psi_1(x,y,z)$$

$$Q = \psi_2(x,y,z),$$

where ϕ and ψ are rational fractions with first order terms taken with respect to the above quantities. These relations are reciprocal. Equation (69) is transformed to $\Phi(X,Y,Z,P,Q) = 0$. If an integral surface of (69) is a plane, then there corresponds to it a multiplicity formed by a point and the planes passing through this point; the equation $\Phi = 0$ is satisfied by the coordinates X,Y,Z of the point for any P and Q. The converse is also true. If a developable surface is an integral surface of equation (69), then the equation $\Phi = 0$ is satisfied by the coordinates X,Y,Z of the points of a curve and by the coefficients P and Q of the tangent planes to this curve and vice-versa.

THE LEGENDRE TRANSFORMATION

In the case of this transformation, we have $X = p$, $Y = q$, $Z = px + qy - z$, $P = x$, $Q = y$ and consequently the equation $\Phi(X,Y,Z) = 0$ corresponds to a Clairault equation

$$\Phi(p,q,px+qy-z) = 0$$

and conversely. On account of what we have just stated, the Lie solutions of $\Phi(X,Y,Z) = 0$ constitute the surface defined by this equation, along with its tangent planes, and the multiplicities defined by the points of this surface assigned to which are the sets of planes which pass through them. To this there corresponds, via the inverse transformation, the singular integral of the Clairut equation and its tangent planes.

The equations whose transformations are linear e.g.

$$A(X,Y,Z)P + B(X,Y,Z)Q + C(X,Y,Z) = 0 \quad , \tag{70}$$

are of the form

$$A(p,q,px+qy-z)x + B(p,q,px+qy-z)y + C(p,q,px+qy-z) = 0 \quad . \tag{71}$$

Now according to the viewpoint adopted by Lie, a linear equation of the kind in (70) admits as its integrals the multiplicities formed by the characteristic curves together with their tangent planes. These multiplicities depend on two parameters; they form a complete generalized integral whose general

ordinary integral surfaces are determined by taking a locus of supports of these multiplicities. To this generalized complete integral there corresponds, in the light of equation (71) a complete integral in the ordinary sense which is formed by the developlable surfaces corresponding to the support curves, and conversely. Equation (71) therefore states *the general form of the partial differential equations in two variables admitting a complete integral formed by the developable surfaces.*

279. Example

Let us consider the surfaces S having the following property: let Γ be the section of a surface S through an arbitrary plane passing through the line $\Delta(z=a,\ x=0)$ and let Λ be the orthogonal trajectories of the lines Γ on S; we assume that the lines Λ are spherical lines lying on the spheres with center 0.

If dx, dy, dz are the differentials of x,y,z on Λ at a point M of S, and $\delta x, \delta y, \delta z$ the differentials on Γ, then we have the following conditions (with respect to rectangular axes)

$$dx\delta x + dy\delta y + dz\delta z = 0, \qquad dz = pdx + qdy,$$
$$\delta z = p\delta x + q\delta y\ , \qquad (z-a)\delta x - x\delta z = 0,$$
$$xdx + ydy + zdz = 0\ .$$

On eliminating $\delta x, \delta y, \delta z$ from the three equations containing them, we have

$$qxdx + (z-a-px)dy + q(z-a)dz = 0$$

and on eliminating dx, dy, dz from this equation and from the two remaining equations, we obtain the partial differential equation of the surfaces S

$$(z-a-px-qy)(x+pz) + aq(py-qx) = 0\ . \tag{72}$$

We may obtain *a priori* a complete integral by observing that the cones of revolution whose vertex is on Δ and whose axis passes through the origin, are integral surfaces depending on 2 parameters. The equation of these cones is

$$x^2 + (y-\lambda)^2 + (z-a)^2 - \mu[\lambda(y-\lambda) + a(z-a)]^2 = 0\ . \tag{73}$$

The characteristic curves are defined by this equation and by

$$2(y-\lambda) + 2\mu(y-2\lambda)[\lambda(y-\lambda) + a(z-a)] + v[\lambda(y-\lambda) + a(z-a)]^2 = 0\ .$$

Equation (72) is of the type (71) since there exists a family of complete integrals formed by the developable surfaces. To see this, we express it in the form

$$p(z-a-px-qy)(z-px-qy) + x((1+p^2)(z-a-px-qy)-aq^2) + ypq(z-px-qy) = 0 \quad .$$

The Legendre transformation then yields a linear equation that is

$$P[a(X^2+Y^2+1) + Z(X^2+1)] + QXYZ - XZ(Z+a) = 0 \quad .$$

The characteristics of this linear equation are determined by the associated system

$$\frac{dX}{a(X^2+Y^2+1)+Z(X^2+1)} = \frac{dY}{XYZ} = \frac{dZ}{XZ(Z+a)} \; .$$

The first integrals are found to be

$$\frac{a+Z}{Y}\,, \qquad\qquad \frac{X^2+Y^2+1}{Z^2}$$

and the general integral is given by

$$a + Z = Yf\left(\frac{X^2+Y^2+1}{Z^2}\right) \quad , \tag{74}$$

where f is an arbitrary function. The characteristics $a+Z = \lambda Y$, $X^2+Y^2+1 = \rho Z^2$ are conics which are the pclar inversions of the cones (73) with respect to the paraboloid $x^2+y^2 = 2z$, the generator of the Legendre transformation (I, 134); we have $\rho(1-\mu(\lambda^2+a^2)) + \mu = 0$.

Remarks. I. When we obtain the equation of the integral surfaces of the transformed equation, following a polar inversion, then the tangential equation of the integral surfaces of the equation in question, is effectively known. For example, equation (74) provides the tangential equation of the surfaces S. If X,Y,Z is a point of a surface (74), then the plane having the equation

$$xX + yY - z - Z = 0$$

is tangent to the transformed surface S. When $ux+vy+wz+h = 0$ is taken to be the general equation of a plane, then this plane will be tangent to S when we have

$$\frac{X}{u} = \frac{Y}{v} = \frac{-1}{w} = \frac{-Z}{h} \; .$$

The tangential equation of the surface S, deduced from (74), is

$$aw + h + vf\left(\frac{u^2+v^2+w^2}{h^2}\right).$$

II. The Cauchy problem relative to a given equation and a curve Ω transforms by polar inversion into a problem which entails finding, for the transformed equation, an integral surface tangent to the transformed developable of Ω.

IV. EXTENSIONS OF THE RESULTS TO MORE GENERAL CASES

280. Extension of the Cauchy method

The Cauchy method extends to the case of an equation in n variables x_j, $j=1,2,\dots,n$. On denoting the partial derivative of the unknown function z with respect to x_j by p_j, $j=1,2,\dots,n$, the equation is of the form

$$F(x_1,\dots,x_j,\dots,x_n,z,p_1,p_2,\dots,p_j,\dots,p_n) = 0 \quad . \tag{75}$$

We assume throughout that F admits first and second continuous partial derivatives with respect to $2n+1$ variables x_j, z, p_j and we denote by X_j and P_j the first partial derivatives of F with respect to x_j and p_j, and by Z, the first partial derivative of F with respect to z. We look for solutions

$$z = \phi(x_1,\dots,x_n)$$

admitting first and second continuous partial derivatives; we will denote the second partial derivative of z with respect to x_j and x_k by $p_{j,k}$. A solution defines in the $(n+1)$ dimensional space (x_j,z) an n-dimensional multiplicity which may be called an integral surface, and on this surface S we can define a system of "curves", namely solutions of the differential system

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} \tag{76}$$

where we assume that in $P_1,\dots,P_n$, z and the p_j are replaced by their values on S. It suffices to assume that $P_1,P_2,\dots,P_n$ are not simultaneously zero on S. As we have

$$dz = p_1dx_1 + \cdots + p_ndx_n$$

$$dp_j = p_{j,1}dx_1 + \cdots + p_{j,n}dx_n \qquad j=1,2,\dots,n$$

then the ratios (76) are also equal, along the curves in question, to the ratios

$$\frac{dz}{P_1p_1+\dots+P_np_n}\,, \qquad \frac{dp_j}{P_1p_1+\dots+P_np_{j,n}}\,, \quad j=1,2,\dots,n \quad .$$

But on differentiating the equation $F = 0$, where z and p_j are replaced by ϕ and its derivatives, with respect to the x_j, we also have

$$X_j + Zp_j + P_1 p_{1,j} + \cdots + P_n p_{j,n} = 0 , \qquad j = 1,2,\ldots,n$$

which allows us to eliminate the $p_{j,k}$ in the above ratios. Finally, along the curves on S, we have

$$\frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{P_1p_1+\ldots+P_np_n} = \frac{-dp_1}{X_1+p_1Z} = \cdots$$

$$= \frac{-dp_n}{X_n+p_nZ} \quad . \tag{77}$$

If we now consider the system in (77) with *a priori* $x_1,x_2,\ldots,x_n, z,p_1,\ldots,p_n$ the independent variables, and assume e.g. that P_1 is nonzero, then this system of $2n$ differential equations admits a unique solution defined by the initial conditions x_j^0, z^0, p_j^0, $j = 1,2,\ldots,n$ and provides, in particular, the curves in question on S once this initial condition has been suitably chosen. It follows that, *if there exists integral surfaces, then these surfaces* S *are the loci of integral curves of (77) such that, for example,*

$$x_k = x_k(x_1,x_j^0,z^0,p_j^0), \qquad k = 2,\ldots,n; \qquad j = 1,2,\ldots,n;$$

$$z = z(x_1,x_j^0,z^0,p_j^0) \quad , \tag{78}$$

corresponding to suitable initial conditions, with the exception of the integral surfaces on which we have $P_j = 0$ *simultaneously,* $j = 1,2,\ldots,n$, *which are singular integrals. The curves defined by (78) are the characteristics.*

In order to verify the existence of integral surfaces, we proceed as in no. 273. We construct an integral surface containing an $n-1$ dimensional multiplicity given *a priori* and satisfying certain conditions, *thus amounting to a solution of the Cauchy problem.* We express Λ in the form

$$x_j = \psi_j(v_1,v_2,\ldots,v_{n-1}); \qquad (j = 1,2,\ldots,n)$$

$$z = \psi(v_1,\ldots,v_{n-1}) \ .$$

We assume that the functions ψ and ψ_j have continuous first and second partial derivatives with respect to the parameters v_k, that the linear system in p_j, $j = 1,2,\ldots,n$,

$$\frac{\partial z}{\partial v_k} = \sum_1^n p_j \frac{\partial \psi_j}{\partial v_k}, \qquad k = 1,2,\ldots,n-1 ,$$

admits a nonzero determinant of order $n-1$ and that on augmenting the system with the equation (75), $F = 0$, the total system is solvable with respect to Λ with respect to the p_j. Thus on Λ, we will have

$$p_j = \theta_j(v_1,v_2,\ldots,v_{n-1}) , \qquad j = 1,2,\ldots,n,$$

where the θ_j have continuous partial derivatives. If we denote by du the common values of the ratios (77), then the solution of this system, which takes the values $x_j = \psi_j$, $z = \psi$, $p_j = \theta_j$ for $u = 0$, is well determined for each system of values of v_k, at least in a restricted domain. The characteristics are given by

$$x_j = \chi_j(u,\psi_j,\ldots,\psi_n,\psi,\theta_1,\ldots,\theta_n) \qquad j = 1,2,\ldots,n, \tag{79}$$

$$z = \chi(u,\psi_1,\ldots,\psi_n,\psi,\theta_1,\ldots,\theta_n)$$

and the values of the p_j by

$$p_j = \tau_j(u,\psi_1,\ldots,\psi_n,\theta,\theta_1,\ldots,\theta_n) \qquad j = 1,2,\ldots,n.$$

It is possible to choose the ψ_j and ψ such that the multiplicity defined by the equations (79) is n-dimensional, i.e. defines what we have called a surface. In order to show that this is an integral surface, it is necessary to show that the Pfaffian condition is satisfied. This can be seen directly, as in the case $n = 2$ (no. 273).

The existence of integral surfaces is thus proved and the Cauchy problem is solved with respect to the hypotheses imposed on the multiplicity Λ. *These hypotheses imply that* Λ *is not situated on a singular integral and is not generated by characteristics depending on* $n-2$ *parameters. The solution to the Cauchy problem is unique once the functions* ψ, ψ_j *and* θ_j *are given.*

281. Extension of the Lagrange method

If we consider a family of surfaces in $(n+1)$-dimensional space depending on n parameters a_j,

$$V(x_1,\ldots,x_n,z,a_1,\ldots,a_n) = 0 , \tag{80}$$

then the partial differential equation of these surfaces is obtained by eliminating the parameter a_j from this equation and the equations

$$\frac{\partial V}{\partial x_j} + p_j \frac{\partial V}{\partial z} = 0 , \qquad p_j = \frac{\partial z}{\partial x_j} , \qquad j = 1,2,\ldots,n, \tag{81}$$

which define the partial derivatives of z with respect to the x_j. In general, this elimination only yields a single equation

$$F(x_1,\ldots,x_n,z,p_1,\ldots,p_n) = 0 \quad . \tag{82}$$

Conversely, if such an equation as (82) is given *a priori*, then we say that an equation of the form (80) defines a complete integral of this equation if the function z of $x_1,\ldots,x_n$ obtained by solving (80) with respect to z is an integral of (81) for any parameters $a_1,a_2,\ldots,a_n$ and if this function does not satisfy any other equation. Elimination of $a_1,a_2,\ldots,a_n$ from (80) and (81) can only yield equation (82); the system of equations in (80) and (81) must be compatible when z is replaced by some solution of (82). Finding solutions of (82) once more reduces to finding functions z, $a_1,\ldots,a_n$ of $x_1,x_2,\ldots,x_n$ satisfying the system (80), (81). These functions satisfy (80) and hence also the equations obtained by differentiation with respect to the x_j,

$$\frac{\partial V}{\partial x_j} + p_j \frac{\partial V}{\partial z} + \frac{\partial V}{\partial a_1}\frac{\partial a_1}{\partial x_j} + \cdots + \frac{\partial V}{\partial a_n}\frac{\partial a_n}{\partial x_j} = 0 , \quad j = 1,2,\ldots,n . \tag{83}$$

These derivations are possible if the a_j possess derivatives, which happens to be the case when n equations of the system (80), (81) are normally solvable with respect to the a_j and if V has second order partial derivatives. On taking account of the relations in (81), the equations in (83) reduce to

$$\frac{\partial V}{\partial a_1}\frac{\partial a_1}{\partial x_1} + \cdots + \frac{\partial V}{\partial a_n}\frac{\partial a_n}{\partial x_j} = 0 , \qquad j = 1,2,\ldots,n. \tag{84}$$

The system of equations in (80), (81) can be replaced by the equivalent system formed by equations (80) and (84). Now the equations in (84) which are linear and homogeneous with respect to the derivatives of V, can only be satisfied when we have

$$V = 0 , \quad \frac{\partial V}{\partial a_1} = 0,\ldots, \frac{\partial V}{\partial a_n} = 0 \quad , \tag{85}$$

or

$$\frac{D(a_1,a_2,\ldots,a_n)}{D(x_1,x_2,\ldots,x_n)} = 0 \quad . \tag{86}$$

If there exists a surface defined by the conditions in (85), then it is a singular integral, this does not exist in general. Condition (86) indicates that the a_j are constrained by one or more relations (I, 126). For example,

suppose we have

$$\frac{D(a_1,a_2,\ldots,a_k)}{D(x_1,x_2,\ldots,x_k)} \neq 0 \quad , \tag{87}$$

whilst the functional determinants of order $k+1$ are zero, then in a particular domain

$$a_{k+1} = w_{k+1}(a_1,\ldots,a_k),\ldots,a_n = w_n(a_1,\ldots,a_k) \ , \tag{88}$$

the system in (84) becomes

$$\sum_1^k \left[\frac{\partial V}{\partial a_m} + \frac{\partial V}{\partial a_{k+1}}\frac{\partial w_{k+1}}{\partial a_m} + \cdots + \frac{\partial V}{\partial a_n}\frac{\partial w_n}{\partial a_n}\right]\frac{\partial a_m}{\partial x_j} = 0 \ , \qquad j = 1,2,\ldots,n,$$

and condition (87) implies

$$\frac{\partial V}{\partial a_m} + \frac{\partial V}{\partial a_{k+1}}\frac{\partial w_{k+1}}{\partial a_m} + \cdots + \frac{\partial V}{\partial a_n}\frac{\partial w_n}{\partial a_m} = 0 \ , \qquad m = 1,2,\ldots,k \ .$$

Consequently:

The integral surfaces will be obtained by assuming that k *of the* a_j *are independent, the remaining* $n-k$ *are arbitrary (differentiable) functions of these* k *first* a_j *and, by eliminating the latter from equations (80) and (89), we can say that a* k*-parameter envelope of the surfaces in (80) can be considered. We will assign to* k *the values* $1,2,\ldots,n-1$. *If there exists an* n*-parameter envelope, then it is a singular integral.*

282. Finding a complete integral. The Poisson identity. The method of Jacobi.

The determination of a complete integral was taken up by Jacobi. We shall confine ourselves to an exposition of his method.

I. We can restrict our attention to an equation $F(x_1,x_2,\ldots,x_n,p_1,\ldots,p_n) = 0$ not containing the unknown function z. Effectively, if we have an equation $\Phi(x_1,x_2,\ldots,z,p_1,\ldots,p_n) = 0$, we may find its solutions in the implicit form, $Z(x_1,\ldots,x_n,z) = 0$. We shall have

$$\frac{\partial Z}{\partial x_j} + p_j\frac{\partial Z}{\partial z} = 0 \qquad ,$$

and Z will be a solution of

$$\Phi\left(x_1,\ldots,z,-\frac{\frac{\partial Z}{\partial x_1}}{\frac{\partial Z}{\partial z}},\ldots,-\frac{\frac{\partial Z}{\partial x_n}}{\frac{\partial Z}{\partial z}}\right) = 0 \quad ,$$

and equation which does not contain Z. We shall therefore consider, exclusively, equations of the form

$$F(x_1,\ldots,x_n,p_1,\ldots,p_n) = 0 \quad . \tag{90}$$

II. *If the* n *functions* $F_j(x_1,\ldots,x_n,p_1,\ldots,p_n)$ *have first continuous partial derivatives with respect to* x_j *and* p_j *if*

$$\Delta \equiv \frac{D(F_1,\ldots,F_n)}{D(p_1,\ldots,p_n)} \neq 0 \quad ,$$

and if the Poisson brackets

$$(F_j,F_k) = \sum_{m=1}^{n}\left(\frac{\partial F_j}{\partial p_m}\frac{\partial F_k}{\partial x_m} - \frac{\partial F_k}{\partial p_m}\frac{\partial F_j}{\partial x_m}\right)$$

are all identically zero, then the expression $dz = p_1dx_1 + \ldots + p_ndx_n$, *where* $p_1,p_2,\ldots,p_n$ *are the values extracted from the system of equations*

$$F_1 = a_1,\ldots,F_n = a_n \quad ,$$

and where the a_j *are constants, is a total differential.*

We must show that $\partial p_j/\partial x_k \equiv \partial p_k/\partial x_j$, for any k and j. These derivatives are given by

$$\frac{\partial F_j}{\partial x_m} + \sum_{q=1}^{n}\frac{\partial F_j}{\partial p_q}\frac{\partial p_q}{\partial x_m} = 0 \quad .$$

From this we deduce

$$\sum_{m=1}^{n}\frac{\partial F_j}{\partial x_m}\frac{\partial F_k}{\partial p_m} + \sum_{m=1}^{n}\sum_{q=1}^{n}\frac{\partial F_j}{\partial p_q}\frac{\partial F_n}{\partial p_m}\frac{\partial p_2}{\partial x_m} = 0 \quad ,$$

and on permuting j and k and then subtracting we see that

$$(F_j,F_k) = \sum_{m=1}^{n}\sum_{q=1}^{n}\frac{\partial F_j}{\partial p_q}\frac{\partial F_k}{\partial p_m}\left(\frac{\partial p_q}{\partial x_m} - \frac{\partial p_m}{\partial x_q}\right) \equiv 0 \quad .$$

Since $\Delta \neq 0$, the n homogeneous equations

$$\sum_{m=1}^{n} \frac{\partial F_k}{\partial p_m} \left(\sum_{q=1}^{n} \frac{\partial F_j}{\partial p_q} \left(\frac{\partial p_q}{\partial x_m} - \frac{\partial p_m}{\partial x_q} \right) \right) \equiv 0 \,, \qquad k = 1,2,\ldots,n,$$

admit the only nonzero solution, and this is for each j. We have

$$\sum_{q=1}^{n} \frac{\partial F_j}{\partial p_q} \left(\frac{\partial p_q}{\partial x_m} - \frac{\partial p_m}{\partial x_q} \right) \equiv 0 \,, \qquad \begin{matrix} j = 1,2,\ldots,n \\ m = 1,2,\ldots,n \;; \end{matrix}$$

and the same argument shows that

$$\frac{\partial p_q}{\partial x_m} \equiv \frac{\partial p_m}{\partial x_q} \,, \qquad \begin{matrix} q = 1,2,\ldots,n \\ m = 1,2,\ldots,n \,. \end{matrix}$$

The proposition is thus established.

III. The Poisson identity. *If the functions* F, G, H *of the* x_j *and* p_j, $j = 1,2,\ldots,n$, *have first and second derivatives, then we have*

$$((F,G),H) + ((G,H),F) + ((H,F),G) \equiv 0.$$

In view of the symmetry, it suffices to show that the terms containing the second order derivatives of F, for example, are zero. These terms arise uniquely from $((F,G),H) + ((H,F),G)$. Now (F,G) for example, is a linear form and is homogeneous with respect to the derivatives of F; if we set

$$X(F) \equiv (F,G) \,, \qquad Y(F) \equiv (F,H) \equiv -(H,F) \,,$$

then we have to consider

$$((F,G),H) + ((H,F),G) \equiv Y(X(F)) - X(Y(F)) \,.$$

Since

$$X(F) = \sum_{1}^{n} \left(\alpha_j \frac{\partial F}{\partial x_j} + \beta_j \frac{\partial F}{\partial p_j} \right)$$

$$Y(F) = \sum_{1}^{n} \left(A_j \frac{\partial F}{\partial x_j} + B_j \frac{\partial F}{\partial p_j} \right) \,,$$

we see straight away that $Y(X(F)) - X(Y(F))$ does not contain any second derivatives of F. Q.E.D.

IV. <u>The Jacobi Method</u>. Given a partial differential equation of the form (90), we obtain a complete integral by adding to this equation the equations

$$F_1 = a_1,\ldots,\qquad F_{n-1} = a_{n-1},\qquad F_j \equiv F_j(x_1,\ldots,x_n,p_1,\ldots,p_n), \tag{91}$$

where the a_j are constants, such that $(F,F_j) = 0$, $(F_j,F_k) = 0$, and that the functional determinant of $F, F_1,\ldots,F_{n-1}$ with respect to $p_1,\ldots,p_n$ is nonzero. Since, on account of Proposition II, with $p_1,\ldots,p_n$ taken from the total system of these equations, the integration of the total differential $dz = p_1dx_1 + \cdots + p_ndx_n$ will provide a function z of $x_1,\ldots,x_n$ depending on $a_1,\ldots,a_{n-1}$ and another arbitrary constant a_n which will be a solution of (90).

In order to determine the functions F_1, F_{n-1}, we proceed recursively. Firstly we can find F_1 to be a solution of the equation $(F,F_1) = 0$, i.e., we consider a first integral of the equations of the characteristics of F. Then we have to determine F_2 by means of the two conditions $(F,F_2) = 0$, $(F_1,F_2) = 0$, i.e. we have to find a common solution to two linear and homogeneous partial differential equations. More generally, matters are reduced to the following problem:

Given r *distinct functions* $F_1(x_1,\ldots,x_n,p_1,\ldots,p_n),\ldots,F_r(x_1,\ldots,x_n,p_1,\ldots,p_n)$, $r<n$, *such that* $(F_j,F_k) \equiv 0$; $j = 1,2,\ldots,r$, $k = 1,2,\ldots,r$ *find a function* Φ *distinct from these and such that* $(F_j,\Phi) = 0$, $j = 1,2,\ldots,r$.

If we put $X_j(\Phi) \equiv (F_j,\Phi)$, then the Poisson identity

$$((F_j,F_k),\Phi) + ((F_k,\Phi),F_j) + ((\Phi,F_j),F_k) \equiv 0 \quad ,$$

in which the first term is zero by hypothesis, can be written as

$$X_k(X_j(\Phi)) - X_j(X_k(\Phi)) \equiv 0 \quad . \tag{92}$$

The determination of Φ will be complete once we know how to solve the following problem in which the variables x_j and p_j play the same role:

Find a common solution Φ *to a system of* r *linear, homogeneous partial differential equations* $X_j(\Phi) = 0$, *with* $m>r$ *variables satisfying conditions (92) for any* j *and* k. *Such a system is called a Jacobian system.*

V. <u>Integration of Jacobian systems</u>. Consider a system of r equations

$$X_j(\Phi) \equiv \sum_{q=1}^{m} c_{j,q}(x_1,\ldots,x_m)\frac{\partial\Phi}{\partial x_q} \quad , \qquad j = 1,2,\ldots,r \tag{93}$$

with $m>r$ variables. In this Jacobian system, the identities (92) are verified. *A transformation of the variables with nonzero functional determinant transforms this system into another Jacobian system.* Since each equation is

transformed into a linear homogeneous equation; $X_j(\Phi) = 0$ becomes $Y_j(\Phi) = 0$ with $y_1,\ldots,y_m$ the new variables and by means of the formulae of the transformation, we have $X_j(\Phi) \equiv Y_j(\Phi)$ where Φ is expressed as a function of the y_j in the second member. Thus, in terms of the formulae of the transformation, we therefore have $X_k(X_j(\Phi)) \equiv Y_k(Y_j(\Phi))$ which proves that the identities in (92) are preserved.

In order to find the common solutions to equations of the system (93), we proceed by successive reductions. Let $y_1, y_2,\ldots,y_m$ be m functions of $x_1,\ldots,x_m$, where $y_2,\ldots,y_m$ are solutions cf the first equation $X_1(\Phi) = 0$ and such that their functional determinant with respect to the x_j is nonzero. Let us take $y_1,\ldots,y_m$ as the new variables. The new system thus obtained will again be a Jacobian system, but since $X_1(y_j) = 0$, $j = 2,\ldots,m$, the first equation will reduce to

$$Y_1(\Phi) \equiv X_1(y_1)\frac{\partial\Phi}{\partial y_1} = 0 \; ; \tag{94}$$

the remaining equations are of the form

$$Y_j(\Phi) = C_{j,1}\frac{\partial\Phi}{\partial y_1} + C_{j,2}\frac{\partial\Phi}{\partial y_2} + \ldots + C_{j,m}\frac{\partial\Phi}{\partial y_m} = 0, \quad j = 2,\ldots \tag{95}$$

The condition $Y_k(Y_j(\Phi)) - Y_j(Y_k(\Phi)) \equiv 0$ is interpreted as the identities

$$\sum_{s=1}^{m}\left[C_{k,s}\frac{\partial C_{j,q}}{\partial y_s} - C_{j,s}\frac{\partial C_{k,q}}{\partial y_s}\right] \equiv 0 \;, \qquad \begin{array}{l} j = 1,\ldots,r \\ k = 1,\ldots,r \\ q = 1,2,\ldots,m \;. \end{array}$$

By taking $j = 1$, $q > 1$, $k > 1$, we obtain

$$\frac{\partial C_{k,q}}{\partial y_1} = 0 \quad ,$$

hence

$$\sum_{s=2}^{m}\left[C_{k,s}\frac{\partial C_{j,q}}{\partial y_s} - C_{j,s}\frac{\partial C_{k,q}}{\partial y_s}\right] \equiv 0 \;, \qquad \begin{array}{l} j = 2,\ldots,r, \\ k = 2,\ldots,r, \\ q = 2,\ldots,m. \end{array}$$

It follows that, by suppressing the first terms in the equations of the system in (95), we obtain a system of $r-1$ equations in the $m-1$ variables $y_2,\ldots,y_m$ that is again of the Jacobian type. Every solution of this reduced system will also satisfy equations (94) and (95) since it does not depend on y_1. By repeating this operation $r-2$ times, we will arrive at a single equation that can be integrated, and on transforming inversely, we will obtain the solution of the system in (93).

283. An application to mechanics

The canonical equations of no. 203 which provide the extremals of the Hamiltonian action are

$$\frac{dy_j}{dx} = \frac{\partial H}{\partial p_j} , \qquad \frac{dp_j}{dx} = - \frac{\partial H}{\partial y_j} , \qquad j = 1,2,\dots,n . \tag{96}$$

They can be regarded as the equations of the characteristics of the partial differential equation (which does not contain the unknown function V),

$$\frac{\partial V}{\partial x} + H(x,y_1,\dots,y_n,p_1,\dots,p_n) = 0 , \qquad p_j = \frac{\partial V}{\partial y_j} .$$

Once we know a complete integral we can write it in the resolved form and it is

$$V = V(x,y_1,\dots,y_n,a_1,\dots,a_n) + a_{n+1} ,$$

where the a_j are constants, since if V is a solution, $V + \text{const.}$ is also.

As in the case of two variables we can show that knowing the complete integral implies knowing the characteristic curves, which are here given by

$$\frac{\partial V}{\partial a_1} = b,\dots, \qquad \frac{\partial V}{\partial a_n} = b_n ,$$

where the a_j are arbitrary constants. These equalities, along with the equalities $p_j = \partial V/\partial y_j$, $j = 1,2,\dots,n$ provide the integrals of the differential system (96).

We refer the reader to the various texts on mechanics, to *Leçons sur les invariant intégraux* by E. Cartan, along with *Leçons sur les equations aux dérivées partielles du premier ordre* and to *Leçons sur le problème de Pfaff* by Goursat.

CHAPTER XVI

SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS OF TWO VARIABLES

As we have already seen, second order partial differential equations arise in studying the theory of surfaces, but above all, they arise in numerous problems of mathematical physics. Indeed, their study lies at the heart of a vast area of reasearch. Here, we shall restrict matters to a discussion of those problems relating to the more elementary examples in two variables. All of these equations can be seen as special cases of the type studied by Monge (1784) and Ampère (1820). The methods relating to the calculus of limits enabled Cauchy (1842) to prove the existence of solutions in the analytic case; his findings were simplified by Darboux and Sophie Kowaleska (1875) and by Goursat (1898). The linear equations can be divided into three types: hyperbolic, elliptic and parabolic. Certain equations of the hyperbolic type can be integrated by a cascading technique due to Laplace (1773); in 1860, Riemann introduced a general technique for simplifying the problem which in some cases can be reduced to quadratures; Picard, in 1890, demonstrated that his technique of successive approximations does, in fact, apply to these equations. The most simple elliptic equation is the Laplace equation defining the harmonic functions which, in turn, anticipate the Dirichlet problem which is based on the question of conformal representation, as is well known. We shall discuss some of the results from this aspect of the theory, in particular Harnack's theorem (1887) and the alternative approach of Schwarz (1870), but we shall not embark on those methods based on the theory of integral equations. For an example of an equation of the parabolic type, we shall consider the heat equation as studied by Fourier and Poisson (1835) whose solutions (non-analytic) were studied by Holmgren (1907).

1. EXISTENCE THEOREMS - CHARACTERISTICS

284. The theorem of Cauchy-Kowaleska

Let us consider a second order partial differential equation

$$F(x,y,z,p,q,r,s,t) = 0 \quad , \tag{1}$$

where p and q are the first partial derivatives, r,s,t the second partial derivatives of the unknown function $z = \phi(x,y)$ (in the notation of Monge). We assume that the function F is analytic about a system of values x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0 of eight variables occurring in it, and that the equation is satisfied for this system of values. We can reduce matters to the case where these eight initial values are taken to be zero: we replace x and y by $x_0 + X$, $y_0 + Y$ and then set

$$z = z_0 + p_0X + q_0Y + \frac{1}{2}\,(r_0X^2 + 2s_0XY + t_0Y^2) + Z \;.$$

The transformed function of F will be analytic about the origin. We can also assume that equation (1) is satisfied for $x = y = z = p = r = q = r = s = t = 0$ and that F is analytic about this point. We shall then assume that the partial derivative of F with respect to r is nonzero at the origin; we then solve the equation with respect to r and write it in the form

$$r = f(x,y,z,p,q,s,t) \;, \tag{2}$$

where the function f *is zero at the origin and analytic in a domain containing this point.*

Theorem. *Given* $\psi(y)$ *and* $\theta(y)$ *analytic at the origin, such that* $\psi(0) = \theta(0) = \psi'(0) = \theta'(0) = 0$, *equation* (2) *admits a solution* $z = \phi(x,y)$ *which is analytic for* $|y|$ *and* $|x|$ *sufficiently small and which satisfies the conditions* $\phi(0,y) \equiv \psi(y)$, $\partial\phi/\partial x\,(0,y) \equiv \theta(y)$ *when* $|y|$ *is sufficiently small.*

To establish this, we can first of all restrict our attention to the case where $\psi(y)$ and $\theta(y)$ are identically zero, by setting $z = \psi(y) + x\theta(y) + Z$. Let us assume then $\psi(y) \equiv \theta(y) \equiv 0$. We need to prove the existence of an analytic solution $z = \phi(x,y)$ which is unique and zero along with its derivative p for $x = 0$.

We intend using the calculus of limits on replacing equation (2) by the system of first order equations in the three unknown functions z,p,q,

$$\frac{\partial z}{\partial x} = p \;, \qquad \frac{\partial p}{\partial x} = f\!\left(x,y,z,p,q,\frac{\partial p}{\partial y}\,,\frac{\partial q}{\partial y}\right), \qquad \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \;. \tag{3}$$

For $x = 0$, we must have $z \equiv 0$, $p \equiv 0$, $q \equiv 0$ and the third equation in (3) implies

$$q = \frac{\partial z}{\partial y} \;;$$

the system in (3) is equivalent to equation (2). When we look for solutions of (3) in the form of power series

$$z = \sum a_{m,n}x^my^n, \quad p = \sum b_{m,n}x^my^n, \quad q = \sum c_{m,n}x^my^n \;, \tag{4}$$

then we see that the coefficients are obtained iteratively by the equalities

$$(m+1)a_{m+1,n} = b_{m,n}, \qquad (m+1)c_{m+1,n} = (n+1)b_{m,n+1}$$

$$(m+1)b_{m+1,n} = P(a_{j,k}, b_{j,k}, c_{j,k}, d_{j,k}, \dots) \ ,$$

where P is a polynomial with positive coefficients: the $d_{j,k},\dots$ are the coefficients of f; the $a_{j,k}$ which occur in P are such that $j+k \leq m+n$; and concerning the $b_{j,k}$ and $c_{j,k}$, we have either $j+k \leq m+n$, or $j=m$, $k=n+1$. But since p and q are zero for $x=0$, $a_{0,\mu}=0$, we have $b_{0,\mu}=0$, $c_{0,\mu}=0$ for any μ; we could then obtain the c and b as we did for the a, corresponding to $m+n = N+1$ by commencing from those corresponding to $m+n = N$. We firstly obtain the c and b corresponding to $m=1$, $n=N$, then knowing these, we proceed to calculate those corresponding to $m=2$, $n=N-1$, etc.

It follows that if we replace the second members of equations (3) by majorant functions and then go about the same calculation, then the coefficients $a,b,c,$ are replaced by positive numbers at least equal to their absolute value. The convergence of the series obtained in this new calculation involves that of the series in (4); these in turn will define a solution of the system (3). As for the common majorants of the second members of (3) we have an expression of the form

$$\omega = \frac{M}{(1-\frac{X+Y+Z+P+Q}{\rho})\left(1-\frac{\frac{\partial P}{\partial Y}+\frac{\partial Q}{\partial Y}}{\rho'}\right)} - M \ .$$

Here, ρ and ρ' are numbers less than the radii of convergence associated to f relative to the variables x,y,z,p,q respectively, on one hand, and $\partial p/\partial y$, $\partial q/\partial y$ on the other; M is taken to be greater than ρ and ρ' and the sum of the moduli of the terms of f for $|x| = \dots = |q| = \rho$ and $\left|\frac{\partial p}{\partial y}\right| = \left|\frac{\partial q}{\partial y}\right| = \rho'$. The limiting equations are

$$\frac{\partial Z}{\partial X} = \frac{\partial P}{\partial X} = \frac{\partial Q}{\partial X} = \omega \ .$$

Since Z,P,Q are zero for $X=0$, it follows that $Z \equiv P \equiv Q$ and matters are reduced to the single equation

$$\frac{\partial Z}{\partial X} = \frac{M}{(1-\frac{X+Y+3Z}{\rho})\left(1-\frac{2\frac{\partial Z}{\partial Y}}{\rho'}\right)} - M \ . \tag{5}$$

We have seen in no. 273 that such an equation admet a unique solution which is analytic and satisfies the conditions of the Cauchy problem $Z=0$ for $X=0$,

since the line thus defined is not a generator of the elementary cones whose vertices are these points. The system in (3) thus admits a unique analytic solution satisfying the above conditions and this proves the theorem.

Remark. We could prove directly the existence of the analytic solution of equation (5) which becomes zero for $X = 0$. To begin with, we majorize the second member by replacing X by kX, $k > 1$, and majorize the solution by no longer subjecting it to the condition of being zero for $X = 0$, but by now supposing that for $X = 0$, it is given by a series with positive coefficients. Then for Z we look for a series in $u = kX + Y$ with positive coefficients; Z must then satisfy the differential equation

$$\left(k \frac{dZ}{du} + M\right)\left(1 - \frac{2}{\rho'} \frac{dZ}{du}\right) = \frac{M\rho}{\rho - u - 3Z} .$$

By taking k sufficiently large in order for the coefficient of dZ/du in the first member to be positive, we see that this equation admits an analytic solution, zero for $u = 0$, whose expansion has positive coefficients.

285. The general existence theorem. Characteristics. The case of linear equations. The classification.

Consider a second order partial differential equation of the general form (1). We may propose a solution of the following *general problem of Cauchy*. We are given *an analytic curve* C *and a series of tangent planes to this curve defining an analytic developable surface containing* C. *We wish to find an integral surface of equation* (1) *containing the curve* C *and tangent along* C *to the developable in question.* The curve C is defined, for example, by $y = g(x)$, $z = h(x)$, where g and h are analytic; we denote the tangent planes by $p = k(x)$, $q = \ell(x)$, where k and ℓ are analytic functions satisfying the contact condition $h'(x) = k(x) + \ell(x)g'(x)$. We return to the initial conditions of the theorem in no. 284 by setting

$$x = Y , \qquad y = g(Y) + X , \qquad z = Z .$$

We have

$$dZ = dz = pdx + qdy = pdY + q[g'(Y)dY + dX]$$

or

$$dZ = [p + qg'(Y)]dY + qdX = PdX + QdY ,$$

and let

$$p = Q - q,g'(Y) = Q - Pg'(Y), \quad q = P$$

where P and Q are the partial derivatives with respect to X and Y, respectively. On denoting the second partial derivatives by R,S,T, we then have

$$\begin{aligned} dp = rdx + sdy &= rdY + s[g'(Y)dY + dX \\ &= dQ - d(Pg'(Y)) \\ &= (SdX + TdY) - g'(Y)(RdX + SdY) - Pg''(Y)dY \end{aligned}$$

$$dq = sdx + tdy = dP = RdX + SdY$$
$$= sdY + t[g'(Y)dY + dX] \quad ,$$

whence we deduce

$$r = T - 2Sg' + g'^2R - Pg'', \qquad s = S - Rg', \qquad t = R \quad .$$

The transformation of equation (1) is obtained by replacing x,y,z,p,q,r,s,t by the values so obtained. For $X = 0$, we must have $Z = h(Y)$ and $P = \ell(Y)$. We shall be covered by the hypotheses of the theorem in no. 284 if we are able to solve with respect to R which will certainly be the case if the partial derivative of the transformed equation with respect to R, is nonzero. Now by stating the terms containing R, the transformed equation is written as

$$F[x,y,z,p,q,g'^2R + \ldots, -Rg' + S,R] = 0 \quad .$$

We must have

$$g'^2 \frac{\partial F}{\partial r} - g' \frac{\partial F}{\partial s} + \frac{\partial F}{\partial t} \neq 0 \quad ,$$

which can be written as

$$\frac{\partial F}{\partial r} dy^2 - \frac{\partial F}{\partial s} dxdy + \frac{\partial F}{\partial t} dx^2 \neq 0 \quad . \tag{6}$$

The Cauchy problem admits a solution, which is unique, if condition (6) is realizable along the given curve C. *Under this* condition x,y,z are the coordinate values of C and p and q the coefficients of the given tangent planes r,s,t are determined by the conditions $dp = rdx + sdy$, $dq = sdx + tdy$, $F = 0$, and by the choice of one of these values at a point.

CHARACTERISTICS

By a *characteristic* we mean *every analytic multiplicity formed by a curve* C, (x,y,z) *and a series of its tangent planes* (p,q) to which we can assign elements r,s,t such that the equations

$$dp = rdx + sdy \ , \qquad dq = sdx + tdy \ ,$$
$$F(x,y,z,p,q,r,s,t) = 0 \quad , \tag{7}$$

$$\frac{\partial F}{\partial r} dy^2 - \frac{\partial F}{\partial s} dxdy + \frac{\partial F}{\partial t} dx^2 = 0 \quad , \tag{8}$$

are compatible. *The Cauchy problem is not solvable by the above method when a characteristic is given.* The curve C is the support of the characteristic.

If S, $z = \phi(x,y)$, is an integral surface, then this surface is a locus of support curves of characteristics, of two different types in general. These are curves of S satisfying the differential equation

$$\frac{\partial F}{\partial r}\,dy^2 - \frac{\partial F}{\partial s}\,dxdy + \frac{\partial F}{\partial t}\,dx^2 = 0 \quad , \tag{9}$$

obtained by assuming that z,p,q,r,s are replaced by their values on S as functions of x and y. Along these curves, the tangent plane to the surface S is the tangent plane of the characteristic. We obtain two families of curves on S, via

$$\left(\frac{\partial F}{\partial s}\right)^2 - 4\,\frac{\partial F}{\partial r}\,\frac{\partial F}{\partial t} \equiv 0 \quad . \tag{10}$$

If the identity (10) holds for arbitrary x,y,z,p,q,r,s,t, then *on every integral surface*, there exists only a single family of characteristics defined by

$$2\,\frac{\partial F}{\partial r}\,dy - \frac{\partial F}{\partial s}\,dx = 0 \quad .$$

The integral surfaces on which equation (9) becomes an identity (surfaces which cannot be derived by the existence theorem), are singular integral surfaces.

The characteristics therefore appear analogous to the characteristics of the first order equations, but do not play the same role in determining the integral surfaces since they cannot be determined *a priori* independently by the integration of a differential system once the integral surfaces are known.

Remark. *Assume that we are given an analytic multiplicity* (x,y,z,p,q) *formed by a curve and a series of its tangent planes.* If there exists an integral surface containing this curve and tangent to these tangent planes, then r,s,t satisfy the equations in (7). We can take, for example, t and s from the first two equations (7) and bring them into the third equation; r will be given by an analytic equation, unless an identity is obtained. If we have an identity, then the derivative of the first member of F, after replacing it by t and s as a function of r, is zero and the condition (8) is realized. When we effectively obtain an equation in r, its finite roots are isolated. Consequently: *when the system in (7) is indeterminate, with* r,s,t *depending on one parameter at least, then the multiplicity in question is a characteristic.*

THE CASE OF LINEAR EQUATIONS

By a linear equation we mean an equation of type (1) whose first member is of the first degree in r,s,t *and whose coefficients in* r,s,t *depend only on* x *and* y. This equation will be of the form

$$A(x,y)r + 2B(x,y)s + C(x,y)t + D(x,y,z,p,q) = 0 \quad . \tag{11}$$

Equation (8) is then

$$A(x,y)dy^2 - 2B(x,y)dxdy + C(x,y)dx^2 = 0 \quad . \tag{12}$$

It can be integrated independently of the integral surfaces which simplifies matters considerably.

CLASSIFICATION

Given a linear, analytic equation *with real coefficients* A,B,C, *uniform*, then we can distinguish the following three cases:

THE FIRST CASE

In a domain Δ *of the plane* Oxy, *we have* $B^2 - AC > 0$. Equation (12) defines in this domain a system of real curves. The characteristics projected along these curves and situated on the real integral surfaces will be real curves. *The equation has (in* Δ*) its characteristics as real, and we say that it is of the hyperbolic type.*

The integral curves of equation (12) have equations of the form

$$\xi(x,y) = a, \qquad \eta(x,y) = b, \qquad a,b \text{ constants},$$

where ξ and η are analytic; through each point of Δ there passes a curve of each family and these curves are not tangents. We can change the variables by setting

$$X = \xi(x,y), \qquad Y = \eta(x,y), \qquad Z = z \ ,$$

and the functional determinant of the transformation is nonzero in Δ. Equation (11) is transformed into a linear equation

$$A_1(X,Y)R + 2B_1(X,Y)S + C_1(X,Y)T + D_1(X,Y,Z,P,Q) = 0 \ ,$$

with

$$A_1(X,Y) = A(x,y)\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B(x,y)\,\frac{\partial\xi}{\partial x}\,\frac{\partial\xi}{\partial y} + C(x,y)\left(\frac{\partial\xi}{\partial y}\right)^2 \ ,$$

$$B_1(X,Y) = A(x,y)\,\frac{\partial\xi}{\partial x}\,\frac{\partial\eta}{\partial x} + B(x,y)\left(\frac{\partial\xi}{\partial x}\,\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\,\frac{\partial\eta}{\partial x}\right) + C(x,y)\,\frac{\partial\xi}{\partial y}\,\frac{\partial\eta}{\partial y} \ ,$$

$$C_1(X,Y) = A(x,y)\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B(x,y)\,\frac{\partial\eta}{\partial x}\,\frac{\partial\eta}{\partial y} + C(x,y)\left(\frac{\partial\eta}{\partial y}\right)^2 \ .$$

$A_1(X,Y)$ and $C_1(X,Y)$ are zero since $\xi = a$ and $\eta = b$ are integral curves of (12). The transformed equation has the reduced form

$$S = D_2(X,Y,Z,P,Q) \ . \tag{13}$$

This reduction is independent of the fact that the solution curves of (12) are characteristics relating to the Cauchy problem. *Every linear equation (11), real, be it analytic or nonanalytic, can be put into the above reduced form by utilizing the integrals of equation (12) provided that* A,B,C

have continuous second derivatives and that $B^2 - AC > 0$. We then say that the integrals of (12) define the characteristics; *when the equation is in the reduced form, we say that it is related to the characteristics.*

THE SECOND CASE

Suppose we have $B^2 - AC \equiv 0$, *then the equation is said to be of the parabolic type* (A,B,C are analytic, $B^2 - AC \equiv 0$ within a domain for which this identity holds as long as A,B,C are defined by analytic continuation). Equation (12), whose first member is a square, defines a single family of curves $\xi(x,y) = a$. If $\eta(x,y) = b$ is the family defined by the condition $B_1(X,Y) = 0$ and if we make the same change of variables as above, then we obtain the reduced form

$$T = D_2(X,Y,Z,P,Q) \quad . \tag{14}$$

Once again, *this reduction is possible even if equation (11) is nonanalytic, just as long as we have* $B^2 - AC = 0$ *in a domain* Δ.

THE THIRD CASE

In a domain Δ, *we have* $B^2 - AC < 0$; *the equation is then of the elliptic type in* Δ. The integral curves of equation (12) are imaginary and, essentially, we must the analyticity of the equation. If $\xi(x,y) = a$ and $\eta(x,y) = b$ are these curves, then we can set

$$\xi(x,y) = X + iY, \qquad \eta(x,y) = X - iY, \qquad Z = z,$$

and from the preceding calculation (we can make the above transformation in the form $\xi(x,y) = u$, $\eta(x,y) = v$, and then set $u = X + iY$, $v = X - iY$) we see that the transformed equation will be

$$R + T = D''(X,Y,Z,P,Q) \quad . \tag{15}$$

With some exceptions (e.g., when A,B,C are constants), this reduction is only possible when the equation is analytic.

Remark. For certain domains we might have $B^2 - AC < 0$ and for others, $B^2 - AC > 0$. These domains are separated by curves for which $B^2 - AC = 0$. These limiting curves are singular integrals of equation (12) on the loci of the singularities of their integrals.

II. MONGE-AMPÈRE EQUATIONS

286. The characteristics of the Monge-Ampère equations. Integral surfaces.

The Monge-Ampère equations are equations of the form

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0 \quad , \tag{16}$$

where H,K,L,M,N are analytic functions of x,y,z,p,q. The equations corresponding to $N\equiv 0$ were studied by Monge, whereas Ampère considered the more general cases.

The characteristics are defined by equation (16) and by

$$Hdy^2 - 2Kdxdy + Ldx^2 + N(tdy^2 + 2sdxdy + rdx^2) = 0 \tag{17}$$

$$dz = pdx + qdy, \qquad dp = rdx + sdy, \qquad dq = sdx + tdy \ . \tag{18}$$

When we take r and t from these last two equations and enter into (16), we obtain (assuming dx and dy to be nonzero)

$$\begin{aligned} &Hdpdy + Ldqdx + Mdxdy + Ndpdq \\ &\quad = s\{Hdy^2 - 2Kdxdy + Ldx^2 + N(tdy^2 + 2sdxdy + rdx^2)\} \quad . \end{aligned}$$

Taking account of (17), there remains

$$Hdydp + Ldxdq + Mdxdy + Ndpdq = 0 \ , \tag{19}$$

and we can see at once that equations (17) and (19) are determined by writing out the system of the last two equations (18) and (16) leaving s indeterminate. This conforms with the remark made in no. 285 concerning the characteristics. Equation (17) is thus written, by replacing $rdx+sdy$ and $sdx+tdy$ by dp and dq

$$Hdy^2 - 2Kdxdy + Ldx^2 + N(dpdx + dqdy) = 0 \quad . \tag{20}$$

The characteristics are determined by the first equation (16) *and by equations* (19) *and* (20). Conversely, (20) implies (17) and r,s,t are indeterminate in view of (19). It is possible to separate the two systems of characteristics. *Firstly, let us assume that* $N\neq 0$. Equations (19) and (20) can be written in the following way: consider dp and dq as variables and note that (19) then represents a hyperbola who asymptotes can easily be described. We obtain

$$(Ndp + Ldx)(Ndq + Hdy) + (NM - LH)dxdy = 0 \ ,$$

$$(Ndp + Ldx)dx + (Ndq + Hdy)dy - 2Kdxdy = 0 \quad .$$

We may assume $Ldx + Ndp = -\lambda dy$, $Hdy + Ndq = -\mu dx$ and the equations become $\lambda\mu = LH - MN$, $\lambda + \mu = -2K$. We then see that the characteristics are given by the equations

$$\text{(I)} \quad \begin{cases} Ndp + Ldx + \lambda dy = 0 \\ Ndq + \mu dx + Hdy = 0 \\ dz = pdx + qdy \end{cases} \qquad\qquad \text{(II)} \quad \begin{cases} Ndp + Ldx + \mu dy = 0 \\ Ndq + \lambda dx + Hdy = 0 \\ dz = pdx + qdy \end{cases}$$

where λ and μ are the roots of the equation

$$\tau^2 + 2K\tau + LH - MN = 0 \quad . \tag{21}$$

The two systems (I) and (II) are distinct, in general, when $K^2 + MN - LH \neq 0$.

If, for example, $dx = 0$ is a solution of the system of equations (16), (17), (18), we have $H + Nt = 0$ and a direct calculation shows that equations (I) and (II) again hold true.

Example (Monge). $N \equiv 0$. The equations of the characteristics are then (if dx and dy are nonzero)

$$\begin{cases} Hdy^2 - 2Kdxdy + Ldx^2 = 0 \,, \\ Hdpdy + Ldqdx + Mdxdy = 0 \,, \end{cases} \tag{22}$$

and in all cases $dz = pdx + qdy$.

If H *is not identically zero*, we can in general solve the first equation with respect to dy and obtain the characteristics in the form

$$dy = \lambda dx, \qquad Hdp + H\mu dq + Mdx = 0 \,, \qquad dz = pdx + qdy \,,$$

where λ and μ are roots of

$$H\tau^2 - 2K\tau + L = 0 \quad .$$

We have two distinct systems when $K^2 - HL \neq 0$. The result is the same when $dy = 0$; we then have $L = 0$. If $dx = 0$, we have $H = 0$ and the first equation must be replaced by $dx = 0$ and the second by $2Kp + Ldq + Mdy = 0$.

If $H \equiv 0$ *but* L *is not identically zero*, then equation (17) reduces to $dx(Ldx - 2Kdy) = 0$ and on account of (16) and (18), we see that the characteristics are given respectively by the two systems

(I) $dx = 0, \quad 2Kdp + Ldq + Mdy = 0, \quad dz = qdy;$

(II) $Ldx - 2Kdy = 0, \quad Ldq + Mdy = 0, \quad dz = pdx + qdy \,.$

Finally, if $H \equiv L \equiv 0$, the equation given is $2Ks + M = 0$. Equation (17) reduces to $dxdy = 0$, and the characteristics which are always distinct, are given by

(I) $dx = 0, \quad dz = qdy, \quad 2Kdp + Mdy = 0;$

(II) $dy = 0, \quad dz = pdx, \quad 2Kdq + Mdx = 0.$

More generally we see that in all cases, the two families of characteristics are only ever identified when $K^2 + MN - LH \equiv 0$.

INTEGRAL SURFACES

We have seen that every integral surface without signature, is in general of two different types, a locus of support of characteristic multiplicities which are tangent to it. Conversely:

Theorem. *If a surface* S *is a locus of support of characteristic multiplicities which are tangent to it, then* S *is an integral surface.*

To see this, if P be a point of S, then there passes a support of characteristics through P: there exist numbers r,s,t satisfying equations (16), (17), (18) where x,y,z,p,q are the coordinates of P and the coefficients of the tangent plane to S at this point; but these numbers r,s,t are indeterminate in such a way that they are subjected to satisfy only the last two equations (18). Putting it another way, every system of numbers r,s,t satisfying the last two equations in (18) satisfy equation (16). Now the second derivatives of z on S at the point P are numbers r,s,t satisfying the last two equations (18); they also satisfy (16). Q.E.D.

CONSEQUENCE

We can proceed to determine integral surfaces using the above theorem. The characteristics taken, for example, in the general form corresponding to N (not identically zero), depend on an arbitrary function. In system (I) we can take $y = (x)$ where Ω is some analytic function, and we have a system of three differential equations for determining z,p,q which in turn introduces three arbitrary constants. But the locus of supports of characteristics so obtained will not in general be surfaces tangent to these characteristic multiplicities.

287. First integrals and intermediate integrals

An analytic function $V(x,y,z,p,q)$ *is said to be a first integral of the system of equations of characteristics* [(I) *e.g., corresponding to the general case* $N \neq 0$] *when this function has a constant value on each characteristic.* This constant value depends on the characteristic. The condition $dV = 0$ must be a consequence of the equations in (I). Now, taking (I) into account, we have

$$dV = \left[\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z} - \frac{L}{N}\,\frac{\partial V}{\partial p} - \frac{\mu}{N}\,\frac{\partial V}{\partial q}\right] dx + \left[\frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial z} - \frac{\lambda}{N}\,\frac{\partial V}{\partial p} - \frac{H}{N}\,\frac{\partial V}{\partial q}\right] dy \quad .$$

This expression must be zero for any dx and dy. It is necessary and sufficient that V satisfies the two conditions

$$\begin{aligned} N\left(\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z}\right) - L\,\frac{\partial V}{\partial p} - \mu\,\frac{\partial V}{\partial q} &= 0 \ , \\ N\left(\frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial z}\right) - \lambda\,\frac{\partial V}{\partial p} - H\,\frac{\partial V}{\partial q} &= 0 \ , \end{aligned} \tag{23}$$

which can be obtained by replacing dx, dy, dp, dq, respectively by

$\frac{\partial V}{\partial p}$, $\frac{\partial V}{\partial q} - \left(\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z}\right)$, $-\left(\frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial z}\right)$ respectively, in the system in (II). We thus obtain analogous results to the Monge example for $N \equiv 0$.

The system in (23) *does not have any solutions in general.* We can confirm the existence of solutions by a reduction method analogous to that described for the Jacobian systems in no. 282. In particular, solutions exist if the system is Jacobian.

Let us assume that V exists. On taking account we will have the identity

$$dV \equiv \frac{\partial V}{\partial z}\,(dz - pdx - qdy) + \frac{1}{N}\,\frac{\partial V}{\partial p}\,(Ndp + Ldx + \lambda dy)$$
$$+ \frac{1}{N}\,\frac{\partial V}{\partial q}\,(Ndq + \mu dx + Hdy) \quad .$$

INTEGRABLE COMBINATIONS

Conversely, if it is possible for us to find an integrable combination of one of the systems of equations of characteristics, (for example, commencing from (I)), i.e., there exists functions ν_1, ν_2, ν_3 of x, y, z, p, q such that

$$\nu_1(dz - pdx - qdy) + \nu_2(Ndp + Ldx + \lambda dy) + \nu_3(Ndq + \mu dx + Hdy)$$

is the total differential of a function $V(x,y,z,p,q)$, then this function is clearly a first integral, since on the characteristics we have $dV = 0$, and necessarily,

$$\frac{\partial V}{\partial z} = \nu_1 \ , \qquad \frac{\partial V}{\partial p} = N\nu_2 \ , \qquad \frac{\partial V}{\partial q} = N\nu_3$$

along with the equations in (23).

AN APPLICATION OF THE FIRST INTEGRALS

Theorem. *If* $V(x,y,z,p,q)$ *is a first integral of one of the systems of equations of the characteristics then the nonsingular integral surfaces of the equation* $V(x,y,z,p,q) = C$, *where* C *is an arbitrary constant, are integral surfaces of the Monge-Ampère equation.*

Effectively, $V(x,y,z,p,q)$ contains p or q and dV is an integrable combination of the equations of the characteristics (I) for example (with $N \neq 0$). The second member of (24) is zero as is the coefficient of $\partial V/\partial z$. If we were to assume for example, that $\partial V/\partial q \neq 0$ and consider on the integral surface S of $V = C$ the multiplicities defined by $Ndp + Ldx + \lambda dy = 0$, then along these multiplicities we also have $Ndq + \mu dx + Hdy = 0$; these will be the characteristics defined by (I).

In accordance with theorem no. 286, S will be an integral surface of the Monge-Ampère equation.

Remarks. I. The characteristics of the equation $V = C$ are defined by

$$\frac{dx}{\frac{\partial V}{\partial p}} = \frac{dy}{\frac{\partial V}{\partial q}} = \frac{dz}{p\,\frac{\partial V}{\partial p} + q\,\frac{\partial V}{\partial q}} = \frac{-dp}{\frac{\partial V}{\partial x} + p\,\frac{\partial V}{\partial z}} = \frac{-dq}{\frac{\partial V}{\partial y} + q\,\frac{\partial V}{\partial z}} .$$

By replacing the denominators of these ratios in the equations in (23), by the numerators, we see that the system in (II) is satisfied. This is another way of proving the theorem and shows that the characteristics (II) of the Monge-Ampère equation situated on the integrals of $V = C$ are the characteristics of this equation $V = C$.

II. Conversely, we can verify that if all the integrals of the first order equation $V(x,y,z,p,q) = C$ are integrals of the Monge-Ampère equation for any constant C, then V is a first integral.

THE INTERMEDIATE INTEGRAL

If we know two distinct first integrals V_1 and V_2 of *one* of the systems of characteristics, i.e. such that one is not a function of the other and if $\psi(V_1,V_2)$ is an arbitrary analytic function of V_1 and V_2, then $\psi(V_1,V_2)$ is again a first integral. Since we have

$$d\psi = \frac{\partial\psi}{\partial V_1}\,dV_1 + \frac{\partial\psi}{\partial V_2}\,dV_2 .$$

All of the nonsingular solutions of $\psi(V_1,V_2) = 0$ are solutions of the Monge-Ampère equation by virtue of the above theorem. Conversely, if S is a locus of support of characteristics tangent to S, then these characteristics belong to the system which admits V_1 and V_2 as first integrals. These characteristics depend analytically on a parameter τ, on each of them; V_1 and V_2 are constants, hence $V_1 = V_2(\tau)$, $V_2 = V_2(\tau)$ where these functions are analytic. By eliminating τ from these relations, we obtain an analytic relation $\psi(V_1,V_2) = 0$ satisfied on S.

As a consequence we have the following:

Theorem. *If we know two distinct first integrals of one of the systems of characteristics,* V_1 *and* V_2 *say, then the general analytic integral of the Monge-Ampère equation is the general integral of the first order equation* $\psi(V_1,V_2) = 0$, *where* ψ *is an arbitrary analytic function. We may write* $\psi = 0$ *in the form* $V_2 = \theta(V_1)$ *and say that this equation defines an intermediate integral of the Monge-Ampère equation.*

288. Examples and applications

I. For the equation $rt - s^2 = 0$, the two systems of characteristics are identified; we have

$$dp = 0, \qquad dq = 0, \qquad dz = pdx + qdy ,$$

and the first integrals p,q and also $z - px - qy$. We have two intermediate integrals

$$p = \psi(q), \qquad z - px - qy = \chi(q) .$$

The developable surfaces may be recovered.

II. Consider the equation $q^2r - 2pqs + p^2t = 0$. The characteristics are identified and are given by

$$dz - pdx - qdy = 0, \qquad pdx + qdy = 0, \qquad qdp - pdq = 0 .$$

We have three linear combinations

$$dz = 0, \qquad d\left(\frac{p}{q}\right) = 0, \qquad d\left(y + x\,\frac{p}{q}\right) = 0 ,$$

hence two intermediate integrals

$$\frac{p}{q} = \psi(z), \qquad y + \frac{xp}{q} = \chi(z) ,$$

which yields

$$\chi(z) = y + x\psi(z) \quad .$$

The integral surfaces are the ruled surfaces whose rectilinear generators are parallel to the plane Oxy.

Remark. In the above two cases, we obtained two intermediate integrals and the characteristics were identified. Darboux showed that the first property necessarily implies the second.

III. The equation $rx^2 - ty^2 = 0$ is linear and can be obtained after the reduction to the canonical form by the methods for the equations of hyperbolic type. But the general method may be applied. The characteristics are given by

$$dz = pdx + qdy, \qquad ydx \pm xdy = 0, \qquad xdp \pm ydq = 0 .$$

By taking the $+$ sign, we obtain the first integrals xy, $z - px - qy$, and matters are reduced to the intermediate integral

$$z - px - qy = f(xy),$$

where f is an arbitrary function. This is a linear equation; the associated system is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-f(x,y)} = \frac{d(xy)}{2xy} \quad .$$

From this we deduce the first integrals to be

$$\frac{y}{x}\ , \qquad \frac{z}{\sqrt{xy}} - F(xy)\ , \qquad \left(F(u) = \int - \frac{f(u)du}{2u\sqrt{u}}\right)$$

where F is an arbitrary function. It follows that

$$z = \sqrt{xy}\ F(xy) + \sqrt{xy}\ G\left(\frac{y}{x}\right)\ ,$$

where G is an arbitrary function, or, on introducing some arbitrary functions,

$$z = \Phi(xy) + x\ \Psi\left(\frac{y}{x}\right)\ .$$

IV. Consider the equation $rt - s^2 + a^2 = 0$, which appears in thermodynamics. The characteristics are given by

$$dz = pdx + qdy, \qquad dp - \varepsilon ady = 0$$

$$dq + \varepsilon adx = 0, \qquad \varepsilon = \pm 1\ .$$

Each system provides an intermediate integral:

$$q + ax = f(p - ay), \qquad q - ax = g(p + ay)\ ,$$

where f and g are arbitrary. Every integral surface is an integral surface of these two equations. Now this system is completely integrable for any f and g. We see this in view of the Poisson brackets or by noting that when we pass through the terms of one of the equations on one side, we obtain a first integral of the equations for the characteristics of the other equation. We reduce matters to a total differential equation. In order to integrate it, we may set

$$p - ay = u, \qquad q + ax = f(u); \qquad p + ay = v, \qquad q - ax = g(v)\ ,$$

which yields

$$p = \frac{u+v}{2}\ , \qquad y = \frac{v-u}{2a}\ , \qquad q = \frac{f(u)+g(v)}{2}\ , \qquad x = \frac{f(u)-g(v)}{2a}\ ,$$

and we have

$$4adz = (u+v)(f'du-g'dv) + (f(u) + g(v))(dv-u)$$

$$= ((u+v)f' - f - g)du + (f + g - g'(u+v))dv\ .$$

By integrating

$$4az = (u+v)(f-g) + 2\int gdv - 2\int fdu$$

or by introducing the primitives of f and g which are arbitrary functions, we have

$$4az = (u+v)[F'(u) - G'(v)] + 2G(v) - 2F(u)\ .$$

Remark. More generally, if two systems of characteristics admit intermediate integrals $V_2 - \theta(V_1)$ and $U_2 - \theta_1(U_1)$, we can see from the relations in (23) and its analogues that $[V_2, U_2] = 0$, hence also that $[\theta(V_1), \theta_1(U_1)] = 0$ and $[V_2 - \theta(V_1),\ U_2 - \theta(U_1)] = 0$. The system of equations $V_2 = \theta(V_1)$, $U_2 = \theta_1(U_1)$ is completely integrable and we calculate as for the preceding example.

V. The Monge-Ampère equation arises when we study surfaces isometric to a surface with a given (metric) ds^2. For further details we refer to Book VII (Tome III) of Darboux's *Théorie des surfaces*.

III. LINEAR EQUATIONS OF HYPERBOLIC TYPE

289. Completely linear equations. The Laplace method.

An equation is said to be completely linear if it is of the form

$$Ar + Bs + Ct + Dp + Eq + Fz + G = 0 \quad , \tag{25}$$

where the coefficients $A,\dots,G$ only depend on x and y. If it happens to be of the hyperbolic type then we can recall the characteristics (no. 285) and put into the reduced form

$$s + ap + bq + cz + d = 0 \quad . \tag{26}$$

a,b,c,d are functions of x,y which are not necessarily assumed to be analytic in this case. *Such an equation is a hyperbolic Laplace equation.* The reduction is possible provided A,B,C are continuous $(B^2 - 4AC > 0)$. We shall assume here that a,b admit continuous first derivatives. We can write equation (26) in the form

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x} + bz\right) + a\left(\frac{\partial z}{\partial x} + bz\right) + \left(c - ab - \frac{\partial b}{\partial y}\right)z + d = 0 \quad .$$

Consequently we achieve the identity

$$k \equiv \frac{\partial b}{\partial y} + ab - c = 0 \quad .$$

We can integrate the equation by replacing it by the system

$$\frac{\partial Z}{\partial y} + aZ + d = 0 \quad , \qquad \frac{\partial z}{\partial x} + bz - Z = 0 \quad ,$$

which is integrated as a differential system: we integrate the first equation with x constant by two quadratures; the constant of integration is a function of x, then we integrate the second with y constant. On permuting the roles of x and y, we can again see the integration when

$$h \equiv \frac{\partial a}{\partial x} + ab - c$$

is identically zero. The functions h and k are called the invariants which do not change when z is replaced by $z\phi(x,y)$ where ϕ is arbitrary.

If each of the invariants is nonzero, we can, for example, set

$$Z = \frac{\partial z}{\partial y} + az$$

which leads us to the equation

$$\frac{\partial Z}{\partial x} + bZ - hz + d = 0 \quad . \tag{27}$$

On withdrawing z from the second equation and bringing it into the first, we obtain the equation

$$\frac{\partial^2 Z}{\partial x \partial y} + a_1 \frac{\partial Z}{\partial x} + b_1 \frac{\partial Z}{\partial y} + c_1 Z + d_1 = 0 \tag{28}$$

with

$$a_1 = a - \frac{1}{h}\frac{\partial h}{\partial y} , \qquad b_1 = b , \qquad c_1 = c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{b}{h}\frac{\partial h}{\partial y} ,$$

$$d_1 = d\left(a - \frac{1}{h}\frac{\partial h}{\partial y}\right) + \frac{\partial d}{\partial y} \quad .$$

The integration of equation (26) is reduced to that of (28), since when (28) is integrated one obtains z in terms of the equation (27). The invariants of equation (28) are

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y} , \qquad k_1 = h \quad .$$

Likewise, when we set

$$Z = \frac{\partial z}{\partial x} + bz \quad , \tag{29}$$

we obtain an equation

$$\frac{\partial^2 Z}{\partial x \partial y} + a_{-1} \frac{\partial Z}{\partial x} + b_{-1} \frac{\partial Z}{\partial y} + c_{-1} Z + d_{-1} = 0 \quad , \tag{30}$$

whose invariants are

$$h_{-1} = k , \qquad k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \quad .$$

It follows that if we denote equation (26) by (E), transformed (28) by (E_1) and transformed (30) by (E_{-1}), then the transformation of (E_1) by the transformation

$$\zeta = \frac{\partial Z}{\partial x} + b_1 Z \quad ,$$

will have invariants h and k, and similarly, the transformation of (E_{-1}) by

$$\zeta = \frac{\partial Z}{\partial y} + a_1 Z$$

will have the invariants h and k of (E). If each of the numbers h_1 and k_{-1} is nonzero, we will then have to repeat the transformation by setting

$$\zeta = \frac{\partial Z}{\partial y} + a_1 Z ,$$

in (28) and

$$\zeta = \frac{\partial Z}{\partial x} + b_{-1} Z$$

in (30).

The transformation of (28), (E_2) say, will have

$$h_2 = 2h_1 - k_1 - \frac{\partial^2 \log h_1}{\partial x \partial y} , \qquad k_2 = h_1$$

as its invariants and the transformation of (30), (E_{-2}) say, will have

$$h_{-2} = k_{-1} , \qquad k_{-2} = 2k_1 - h_{-1} - \frac{\partial^2 \log k_{-1}}{\partial x \partial y}$$

as its invariants; and so on. Equation (26) will be integrated by this *cascade* method when we arrive at an index j for which an invariant h_j or k_{-j} is zero. In order to carry out these operations, we need to assume that a,b,c,d possess the necessary derivatives.

We refer to tome II of Darboux's *Théorie des surfaces* for the more detailed study of this method of Laplace which is covered by a more general method of Darboux.

Example. The equation $(\alpha x + \beta y)s + \gamma p + \delta q = 0$, where $\alpha, \beta, \gamma, \delta$ are constants has as its invariants

$$h = \frac{\gamma(\delta-\alpha)}{(\alpha x+\beta y)^2} , \qquad k = \frac{\delta(\gamma-\beta)}{(\alpha x+\beta y)^2} ;$$

when h or k is zero, we revert to a quadrature. For example, if $\delta = \alpha$, we have

$$z = (\alpha x+\beta y)^{-\gamma/\beta}[\lambda(x) + \int \mu(y)(\alpha x+\beta y)^{\gamma/\beta-1}\, dy]$$

where λ and μ are arbitrary functions. When $hk \neq 0$, we have, for example,

$$h_1 = \frac{(\delta-2\alpha)(\beta+\gamma)}{(\alpha x+\beta y)^2} ,$$

which yields two stages of integration.

Let us take $\beta + \gamma = 0$. We shall find

$$\gamma(\delta-\alpha)z = (\delta-\alpha)[\lambda(x) + \int \mu(y)(\alpha x+\beta y)^{-\delta/\alpha+1}\, dy]$$
$$+ (\alpha x+\beta y)[\lambda'(x) + (\alpha-\delta) \int \mu(y)(\alpha x+\beta y)^{-\delta/\alpha}\, dy] .$$

When $-\delta/\alpha$ is a positive integer m, we can produce a solution without signs of quadrature by replacing the arbitrary function $\mu(y)$ by the derivative of order $m+2$ of an arbitrary function and then integrate by parts.

Remarks. I. When we take the equation of example III of no. 288, $rx^2 - ty^2 = 0$ and consider the characteristics with $xy = \xi$, $y/x = \eta$, then we obtain the equation

$$2\xi \frac{\partial^2 z}{\partial\xi\partial\eta} - \frac{\partial z}{\partial\eta} = 0 ,$$

which appears in the Laplace equations and which is also written as

$$2\xi \frac{\partial}{\partial\xi}\left(\frac{\partial z}{\partial\eta}\right) - \frac{\partial z}{\partial\eta} = 0 . \tag{31}$$

This equation can be integrated straight away and yields

$$z = \sqrt{\xi}\,\lambda(\eta) + \mu(\xi) ,$$

where $\lambda(\eta)$ *is an arbitrary differentiable function and* $\mu(\xi)$ *is some other function.* We recover the result of no. 288 but without the assumption that λ and μ are analytic.

Recalling the variables x,y, we have

$$z = \sqrt{xy}\,\lambda\left(\frac{y}{x}\right) + \mu(xy)$$

and this function satisfies the equation $rx^2 - ty^2$ *when* λ *and* μ *have second derivatives.*

II. In a similar way, the equation of the vibrating strings (I, 92) is of the form $r - a^2t = 0$. With respect to the characteristics by the change of variables $y - ax = \xi$, $y + ax = \eta$, it becomes

$$\frac{\partial^2 z}{\partial\xi\partial\eta} = 0 ,$$

whose general integral is $\lambda(\xi) + \mu(\eta)$, where λ *and* μ *are only once differentiable.* But, in order to obtain solutions of $r - a^2t = 0$, we must take $\lambda(y - ax) + \mu(y + ax)$, *where* λ *and* μ *admit second derivatives.* These observations are quite general and reflect the fact that the existence of one or two second partial derivatives of a function $f(x,y)$ does not imply the existence of the three derivatives.

290. The adjoint equation and the Riemann method.

Given a homogeneous, hyperbolic Laplace equation

$$E(z) \equiv \frac{\partial^2 z}{\partial x\partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0 ,$$

where a, b, c are functions of x and y admitting first partial derivatives and u is a function of x and y admitting first and second partial derivatives, then we can write:

$$u \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(u \frac{\partial z}{\partial x}\right) - \frac{\partial u}{\partial y}\frac{\partial z}{\partial x}$$

$$= \frac{\partial}{\partial y}\left(u \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z \frac{\partial u}{\partial y}\right) + z \frac{\partial^2 u}{\partial x \partial y} ,$$

$$au \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(auz) - z \frac{\partial au}{\partial x} ,$$

$$bu \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(buz) - z \frac{\partial bu}{\partial y} .$$

Consider the expression

$$F(u) \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial au}{\partial x} - \frac{\partial bu}{\partial y} + cu ;$$

we obtain

$$uE(z) - zF(u) \equiv \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y} ,$$

with

$$\begin{aligned} G &= -z \frac{\partial u}{\partial y} + auz , \\ H &= u \frac{\partial z}{\partial x} + buz . \end{aligned} \tag{33}$$

Equation $F(u) = 0$ *is the Riemann adjoint equation of* $E(z) = 0$. It is written by expanding

$$F(u) \equiv \frac{\partial^2 u}{\partial x \partial y} - a \frac{\partial u}{\partial x} - b \frac{\partial u}{\partial y} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) u = 0 .$$

We see at once that there is a reciprocity between these two equations: the adjoint of $F(u) = 0$ is $E(z) = 0$.

It follows from equation (32) that, if we consider a domain Δ in the plane (x,y) bounded by a simple rectifiable curve Γ, then following Riemann's formula, we have

$$\iint_{\Delta} [uE(z) - zF(u)]\, dxdy = \int_{\Gamma^+} Gdy - Hdx ,$$

provided that u and z admit continuous partial derivatives which appear in $E(z)$ and $F(u)$. In particular, if z and u are the solutions of the equations $E(z) = 0$ and $F(u) = 0$ respectively, then we have

$$\int_{\Gamma^+} Gdy - Hdx = 0 . \tag{34}$$

The Riemann method. We propose a solution of the Cauchy problem for the equation $E(z) = 0$, under the following conditions. We take a continuous curve arc AB on which each of the coordinates x,y is a strictly monotone function of the other (increasing or decreasing). Such an arc is rectifiable.

On AB the values of z and its partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ are given; these values assumed to be continuous are bound by the following relation at any point P of AB, we have

$$z(P) = z(A) + \int_A^P \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \quad .$$

We might, for example, take $\frac{\partial z}{\partial x} = \phi(x)$, $\frac{\partial z}{\partial y} = \psi(y)$ on AB and z will be determined by this equation when $x(A)$ is given.

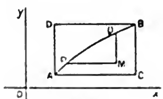

Fig. 85.

The Riemann method provides the solution of equation $E(z)$ satisfying these Cauchy conditions throughout the rectangle whose sides are parallel to the axes (i.e. to the projections of the characteristics) and for which AB is a diagonal. For a point M of this rectangle, the parallels to the axes taken through M intersect the arc AB at P and Q respectively (fig. 85). Let us take x,y to be the coordinate of M and ξ and η the coordinates of a variable point on the contour PMQP formed by MP parallel to Ox, MQ parallel to Oy, and the arc QP of the curve AB (fig. 85).

If we assume that the solution $z(x,y)$ *where* $z(M)$ *considered on* $E(z) = 0$ *exists, and if* u *is a solution of the adjoint equation* $F(u) = 0$ *whose existence is also assumed, then on account of (34) we have*

$$\int_P^M H d\xi - \int_M^Q G d\eta + \int_P^Q G d\eta - H d\xi = 0 \quad .$$

The third integral is a known function of x and y (or of M), since G and H, given by (33), are known completely on the arc PQ. On the other hand, on integrating by parts we have

$$\int_P^M H d\xi = \int_P^M \left(buz + u\frac{\partial z}{\partial \xi}\right) d\xi = (uz)_P^M + \int z\left(bu - \frac{\partial u}{\partial \xi}\right) d\xi \quad ,$$

and the formula in (35) becomes

$$(uz)_M - (uz)_P + \int_P^M z\left(bu - \frac{\partial u}{\partial \xi}\right) d\xi - \int_M^Q z\left(au - \frac{\partial u}{\partial \eta}\right) d\eta$$

$$= \int_Q^P G d\eta - H d\xi \quad . \tag{36}$$

We eliminate the integrals of the first member by taking for $u(\xi,\eta)$ a solution of the adjoint equation satisfying the conditions:

$$\frac{\partial}{\partial \xi} u(\xi,y) = b(\xi,y)u(\xi,y) \qquad \text{on} \quad MP$$

$$\frac{\partial}{\partial \eta} u(x,\eta) = a(x,\eta)u(x,\eta) \qquad \text{on} \quad MQ \; .$$

When there exists an integral u of the adjoint equation satisfying these conditions, then it is only defined to within a factor; we may assume $u(M) = 1$ and the above conditions become

$$\begin{aligned} \log u(\xi,y) &= \int_x^\xi b(t,y)\, dt \quad , \\ \log u(x,\eta) &= \int_y^\eta a(x,t)\, dt \quad . \end{aligned} \tag{37}$$

Recalling (36), *we obtain the following result:*

When the solution $z(x,y)$ *exists and it is possible to obtain for the adjoint equation a solution satisfying conditions* (37), *then necessarily we have*

$$z_M = (uz)_P + \int_Q^P z\left(au - \frac{\partial u}{\partial \eta}\right) d\eta - u\left(\frac{\partial z}{\partial \xi} + bz\right) d\xi \quad . \tag{38}$$

This formula of Riemann shows that if the desired solution exists, then it is unique.

THE CASE OF THE NON-HOMOGENEOUS EQUATION

If the equation is $E(z) = f(x,y)$, where $f(x,y)$ is given, then we can again define the solution of the adjoint equation $F(u) = 0$ by the conditions (37). The double integral in Δ of the expression (32) will no longer be zero but equal to

$$\iint_\Delta uf(x,y)\, dxdy \quad ; \tag{39}$$

the first member of (34) will be equal to this integral. The expression in the first member of (35), which is the first member of (34) with sign changed in the case of figure 85 (i.e. when y increases along AB in relationship to x), is therefore equal to the expression (39) with sign changed. When y decreases as x increases along AB, then it is necessary to change this sign. The solution is then given by (38), to the second member of which the expression (39) is added on multiplication by ± 1. It is necessarily unique.

Remark. In applications of the relationship (38), the function u which depends on the current coordinates ξ,η and the x,y coordinates of M, will be known explicitly (we call such a function a Riemann function). We will then be able to find this particular solution of the adjoint equation. In simple cases, we can show directly that (38) provides a solution to the equation in question, with the prevailing conditions satisfied, but the existence theorem which will be stated in no. 292 makes such a verification redundant.

291. Applications

I. THE EQUATION OF VIBRATING STRINGS

Such an equation is taken to be of the form

$$\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = 0 \quad , \tag{40}$$

where t denotes time and x the abscissa of a point on the string (see I, 92). For $t=0$ we have $y=\phi(x)$, $0 \leqslant x \leqslant L$. Let and $\partial y/\partial t = \psi(x)$, $0 \leqslant x \leqslant L$, these functions being zero for $x=0$ and $x=L$. To these *initial conditions* we also consider *the boundary conditions*, $y=0$ and $\partial y/\partial t = 0$ for $x=0$ and $x=L$, for any t. In view of the equation we must assume that $\phi''(x)$ exists.

When we consider the equation of the characteristics $x-at = \text{const.}$, $x+at = \text{const.}$, we find, on account of Remark II in no. 289, that the general integral is of the form $y = \lambda(x-at) + \mu(x+at)$ and we must assume that λ and μ have second derivatives. The function $\psi(x)$ must have a derivative. For any t, the boundary conditions, yield

$$\lambda(-at) + \mu(at) = 0, \qquad \lambda'(-at) - \mu'(at) = 0 \;;$$

$$\lambda(L-at) + \mu(L+at) = 0, \qquad \lambda'(L-at) - \mu'(L+at) = 0 \;;$$

it follows that the sought after solution, if it exists, is defined for any x and t. The functions λ and μ are periodic of period $2L$ since $\lambda(2L+v) = -\mu(-v) = \lambda(v)$. For $t=0$, we will have, for all x, $y = \lambda(x)+\mu(x)$ periodic with period $2L$ and $\lambda(-x) + \mu(-x) = -[\mu(x) + \lambda(x)]$. Thus, if the

desired solution exists, then for $t=0$, it is equal to a function $\Phi(x)$, which is odd, of period $2L$, and which coincides with $\phi(x)$ for $0 \leq x \leq L$. Similarly, for $t=0$, $\partial y/\partial t$ is equal to a function $\Psi(x)$ of period $2L$, which is even and which coincides with $\psi(x)$ for $0 \leq x \leq L$.

Regarding equation (40) with respect to its characteristics, we obtain an equation of the form

$$\frac{\partial^2 y}{\partial X \partial T} = 0 ,$$

which coincides with its adjoint; the Riemann function u is equal to the constant 1. The relationship in (38) yields the value of the integral in question, that is if it exists. We obtain

$$y(x,t) = \Phi(x-at) + \frac{1}{2}\int_{x-at}^{x+at} \left(\Phi'(v) + \frac{\Psi(v)}{a}\right) dv$$

or

$$y = \frac{1}{2}\left[\Phi(x+at) + \Phi(x-at)\right] + \int_{x-at}^{x+at} \Psi(v)\, dv \quad . \tag{41}$$

It is now clear that we indeed have a solution satisfying the conditions imposed for $0 \leq x \leq L$ *and* $t \geq 0$ *when* Φ *has a second derivative and* Ψ *a first derivative.* We shall compare this result with that obtained by the Bernoulli method (I, 92). The relationship (41) (*which is due to d'Alembert*) can in fact be obtained independently of the Riemann method.

II. HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

If the coefficients a,b,c of a homogeneous equation $E(z)=0$ are constant, then the equation can be simplified by changing z to ze^{-bx-ay} which clears terms containing first order derivatives. Every homogeneous equation with constant coefficients can be reduced to the form

$$\frac{\partial^2 z}{\partial x \partial y} + cz = 0 \quad , \tag{42}$$

where c is a constant. This coincides with its adjoint.

The Riemann function in this case must be a solution of equation (42) which is equal to 1 for $\xi = x$ and $\eta = y$, since the second members of the equalities in (37) are zero. We can find such a solution in the form $u = \Phi(v)$, with $v = (x-\xi)(y-\eta)$, we will have $u = \Phi(0)$ for $\xi = x$ and $\eta = y$, we must take $\Phi(0) = 1$. We have

$$\frac{\partial u}{\partial x} = \Phi'(v)(y-\eta) , \qquad \frac{\partial^2 u}{\partial x \partial y} = \Phi''(v)v + \Phi'(v) \quad ,$$

where $\Phi(v)$ must satisfy the differential equation

$$\Phi''(v)v + \Phi'(v) + c\Phi(v) = 0 \quad .$$

This equation reduces to the Bessel equation (no. 124) by setting $4cv = x^2$, this has an integer solution equal to 1 for $v = 0$,

$$u = (v) = J_0(\sqrt{4cv})$$

$$= 1 - cv + \frac{c^2v^2}{2^2} + \cdots + \frac{(-1)^n c^n v^n}{(n!)^2} + \cdots$$

The function u is thus defined and does not change when x and ξ, y and η are permuted simultaneously.

The telegraphic equation

$$\frac{\partial^2 Z}{\partial X^2} - \frac{\partial^2 Z}{\partial t^2} - 2\frac{\partial Z}{\partial t} = 0 \tag{43}$$

with the initial conditions $Z = f(X)$, $\partial Z/\partial t = g(X)$ for $t = 0$, reduces to equation (42) when the characteristics are considered, by setting $2x = X - t$, $2y = X + t$ and then let $Z = ze^{-t}$; we obtain $c = 1$. The corresponding solutions will be

$$z = f(2x), \qquad \frac{\partial z}{\partial x} = h(2x) \; ,$$

$$h(2x) \equiv f'(2x) - f(2x) - g(2x) \; .$$

The Riemann formula yields

$$z(x,y) = f(2y) - \int_x^y [f(2\xi)\Phi'[(x-\xi)(y-\xi)](\xi-x)$$

$$+ \Phi[(x-\xi)(y-\xi)]h(2\xi)]\, d\xi \quad . \tag{44}$$

This can be expressed more symmetrically by integrating by parts the expression

$$f(2\xi)\Phi'((x-\xi)(y-\xi))\left(\xi - \frac{x+y}{2}\right) \quad ;$$

we obtain

$$z(x,y) = \frac{f(2x)+f(2y)}{2} + \int_x^y \left[f(2\xi)\Phi'(v)\,\frac{x-y}{2}\right.$$

$$\left. + \Phi(v)(f(2\xi) + g(2\xi))\right] d\xi, \qquad c = 1 \quad v = (x-\xi)(y-\xi) \; .$$

Returning to the variables X and t of equation (43) and changing ξ to $\xi/2$ in the integral, we obtain the solution in the form

$$Z = e^{-t}\left\{\frac{f(X+t)+f(X-t)}{2}\right.$$
$$\left.+\int_{X-t}^{X+t}\left[\frac{1}{2}\,\Phi(w)[f(\xi)+g(\xi)]-\frac{t}{4}\,f(\xi)\Phi'(w)\right]d\xi\right\} ,$$

with

$$w = \frac{(X-\xi)^2-t^2}{4}\,, \qquad \Phi(w) = J_0$$

$$\Phi(w) = J_0(\sqrt{4w}) = 1 - w + \frac{w^2}{2} - \cdots + \frac{(-1)^n w^n}{(n!)^2} + \cdots$$

In the guise of (44), it can be at once seen that $z(x,x) = f(2x)$, $\partial z/\partial x\ (x,x) = h(2x)$, and equation (42) is satisfied since $\Phi(v)$ satisfies the differential equation which also implies that $\Phi'''(v)v + 2\Phi''(v) + c\Phi'(v) = 0$. *More directly, we can say that* (44) *provides a solution satisfying the given initial conditions, provided that* f *is differentiable and* h *integrable;* this solution is unique following what was said before. *It follows that the value* Z *is the solution of the telegraphic equation* (43) *provided that* f *admits a second derivative and* h *a first derivative.* In these applications, we assume f and g to be zero, except in some interval which can be assumed to be $(0,\alpha)$ $\alpha > 0$. If $X > \alpha$ is given and if we consider the value of $Z(x,t)$ for t increasing, then we see that Z is zero for as long as $t < X - \alpha$ and that for $t > X$, we obtain

$$Z = e^{-t}\int_0^{\alpha}\left[\frac{1}{2}\,\Phi(w)(f(\xi)+g(\xi)) - \frac{t}{4}\,f(\xi)\Phi'(w)\right]d\xi \;.$$

When t increases indefinitely, Z tends towards zero as a result of the exponential factor. On account of (62) of no. 128, we have, for sufficiently large t, w is negative, $\Phi(w) > 0$ and

$$\Phi(w) = J_0(2\sqrt{w}) \quad \frac{K}{\sqrt{t}}\; e^{t} \quad ,$$

where K is a constant. On the other hand, we have (from no. 125), $J_0'(x) = -J_1(x)$, hence

$$\Phi'(w) = \frac{1}{\sqrt{w}}\,J_0'(2\sqrt{w}) = -\,\frac{1}{\sqrt{w}}\,J_1(2\sqrt{w}) \quad ,$$

which provides for $\Phi'(w)$, the bound

$$|\Phi'(w)| < \frac{K'}{t\sqrt{t}} e^t \quad ,$$

and show that $|Z|$ remains less than $M/\sqrt{t}$, where M is fixed. Moreover, if $f(\xi)$ and $g(\xi)$ retain the same sign between 0 and α, we can easily see that $|Z|$ remains greater than $m/\sqrt{t}$.

Remarks. I. Poincaré outlined an alternative method of deriving the telegraphic equation (*Compte rendus*, 1893). To this and related topics, we refer to Picard's *Leçons sur quelques types simples d'equations aux dérivées partielles*.

II. The asympotic values of $\Phi(w)$ and $\Phi'(w)$, utilized above, can be deduced directly from the series expansion by a similar method to that described for certain entire functions (I, 211), but one that is more explicit.

292. The method of successive approximations

The Cauchy problem posed in no. 290 for the equation

$$\frac{\partial^2 y}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = f(x,y) \quad , \tag{45}$$

where a,b,c are continuous functions of x and y, along with f, was solved by Picard. On the arc A,B we are given the values $\phi(x)$ and $\psi(y)$ of $\partial z/\partial x$ and $\partial z/\partial y$ respectively, which are assumed to be ok. For the first value close to z we take a function $z_0 = z_0(M) = z_0(x,y)$ defined by

$$z_0 = \pm \iint_\delta f(x,y)\, dxdy + z(A) + \int_A^Q \phi(x)\, dx + \int_A^P \psi(y)\, dy \quad ,$$

where δ is the domain bounded by the segments PM, MQ and the arc PQ of AB (fig. 85), and we take the sign $-$ if the arc is as in fig. 85 and the sign $+$ in the opposite case. In the case of the diagram, we will have

$$\frac{\partial z_0}{\partial x} = \phi(x) - \int_M^Q f(x,y)\, dy \quad ;$$

this is seen by giving x a small increment. Then

$$\frac{\partial^2 z}{\partial x \partial y} = f(x,y) \quad ; \tag{46}$$

on AB z_0 is equal to the given value for z and $\partial z_0/\partial x = \phi(x)$. We then proceed to define the functions $z_n(x,y)$ iteratively

$$z_n = \pm \iint_\delta \left(a \frac{\partial z_{n-1}}{\partial x} + b \frac{\partial z_{n-1}}{\partial y} + c z_{n-1} \right) dxdy \ , \quad n = 1,2,\ldots \tag{47}$$

where the sign is taken to be $+$ in the case of fig. 85 and $-$ in the opposite case. z_n is zero along with its first partial derivatives on AB, and we have

$$\frac{\partial^2 z_n}{\partial x \partial y} = -a \frac{\partial z_{n-1}}{\partial x} - b \frac{\partial z_{n-1}}{\partial y} - c z_{n-1}, \qquad n = 1,2,\ldots \tag{48}$$

$$\frac{\partial z_n}{\partial x} = \pm \int_M^Q \left(a \frac{\partial z_{n-1}}{\partial x} + b \frac{\partial z_{n-1}}{\partial y} + c z_{n-1} \right) dy \ , \tag{49}$$

$$\frac{\partial z_n}{\partial y} = \pm \int_M^P \left(a \frac{\partial z_{n-1}}{\partial x} + b \frac{\partial z_{n-1}}{\partial y} + c z_{n-1} \right) dx \ ; \tag{50}$$

the signs $\pm$, which depend on the form AB, are independent of the index n. We now intend to show that the series

$$z_0 + z_1 + \cdots + z_n + \cdots \ , \tag{51}$$

$$\frac{\partial z_0}{\partial x} + \frac{\partial z_1}{\partial x} + \cdots + \frac{\partial z_n}{\partial x} + \cdots \ , \tag{52}$$

$$\frac{\partial z_0}{\partial y} + \frac{\partial z_1}{\partial y} + \cdots + \frac{\partial z_n}{\partial y} + \cdots \ , \tag{53}$$

converges uniformly in the rectangle AB CD.

Since a,b,c are bounded it follows that the series whose general term is the expression (48), also converges uniformly. The sum z of the series (51) will then have first partial derivatives and a second with respect to x and y which will be the sum of the series (52), (53) and the series with general term (48). By adding the equality (46) to those of (48) relative to $n = 1,2,\ldots$, we see that z *will be the solution of equation* (45) *and this solution will satisfy the given initial conditions.*

Let us assume the situation depicted by fig. 85 and take M above AB. Let α,β be the coordinates of A and γ,ε those of B. Let K be the common bound of $|a|$, $|b|$, $|c|$ in the rectangle ABCD and K' the common bound of

$$|z_0|, \quad \left|\frac{\partial z_0}{\partial x}\right| , \quad \left|\frac{\partial z_0}{\partial y}\right| \ .$$

There exists a no. Δ such that, for $n \geqslant 1$, and M above AB, we have

$$|z_n| \leqslant \frac{\Delta^n \rho^n}{n!}, \qquad \left|\frac{\partial z_n}{\partial x}\right| < \frac{\Delta^n \rho^n}{n!}, \qquad \left|\frac{\partial z_n}{\partial y}\right| < \frac{\Delta^n \rho^n}{n!}, \tag{54}$$

with $\rho = \gamma - x + y - \beta$. Since, on account of (49), we have for $n = 1$,

$$\left|\frac{\partial z_1}{\partial x}\right| \leqslant 3K'K \int_Q^M dy = 3KK'QM < 3KK'(y-\beta) < 3KK'(\gamma - x + y - \beta),$$

and a similar result for the derivative with respect to y, whereas (47) gives

$$|z_1| \leqslant 3KK' \iint_\delta dxdy < 3KK'(y-\beta)(\gamma-x) < 3KK'(\gamma-\alpha)\rho .$$

The inequalities will hold for $n = 1$, when $\Delta > 3KK'$, $\Delta > 3KK'(\gamma-\alpha)$. If these inequalities hold for n, then from (49), we have

$$\left|\frac{\partial z_{n+1}}{x}\right| < 3K\Delta^n \int_Q^M \frac{(\gamma-x+y-\beta)^n}{n!} dy < 3K\Delta^n \frac{\rho^{n+1}}{(n+1)!},$$

and the same result for the derivative with respect to y, whereas from (47),

$$|z_{n+1}| < 3K\Delta^n \int_Q^M dy \int_M^P \frac{(\gamma-x+y-\beta)^n}{n!} dx < 3K\Delta^n \int_\beta^y dy \int_x^\gamma \frac{\rho^n}{n!} dx$$

$$< 3K\Delta^n \frac{\rho^{n+2}}{(n+2)!} < 3K(\gamma-\alpha+\varepsilon-\beta)\Delta^n \frac{\rho^{n+1}}{(n+1)!} .$$

It thus suffices to again assume $\Delta > 3K$ and $\Delta > 3K(\gamma-\alpha+\varepsilon-\beta)$ in order for the inequalities (54) to be true. On replacing ρ by its maximum $\gamma - \alpha + \varepsilon - \beta$, we obtain inequalities which are also true for M below AB and which imply uniform convergence in ACBD.

UNIQUENESS

Let us assume that there exists in ABCD another solution Z satisfying the same initial conditions. $Z - z$ would satisfy the homogeneous equation with zero initial conditions, let v be this function. We would have

$$\pm \iint_\delta \frac{\partial^2 v}{\partial x \partial y} dxdy = \mp \iint_\delta \left(a \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + cv\right) dx\, dy$$

and the integral of the first member (with the factor ± 1) gives v. We would have

$$v = \pm \iint_\delta \left(a \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + cv\right) dxdy \ ,$$

$$\frac{\partial v}{\partial x} = \pm \int_M^Q \left(a \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + cv\right) dy$$

$$\frac{\partial v}{\partial y} = \pm \int_M^P \left(a \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + cv\right) dx \quad .$$

On subtracting these equalities from the appropriate side of the equalities (47), (49) and (50) respectively, we see that (47), (49), (50) are again satisfied when z_n and z_{n-1} are replaced by $z_n - v$ and $z_{n-1} - v$. As a result of the above proof we would see that $z_n - v$ satisfies some inequalities similar to (54), with Δ changed, since K' would be replaced by the bound of $|z_0 - v|, \ldots$. Consequently, $z_n - v$ would have a limit 0 and, since z_n tends to zero, v would be identically zero.

As a definitive result, we have PICARD's THEOREM.

Under the sole condition that the functions a,b,c,f *are continuous in the rectangle* ABCD *and that the given values* $\phi(x)$ *and* $\psi(x)$ *of* $\partial z/\partial x$ *and* $\partial z/\partial y$ *are continuous on* AB, *equation* (45) *admits a unique solution in* ABCD, *which is the sum of the series admits a unique solution in* ABCD, *which is the sum of the series* (51), *where* z_0 *and* z_n *are calculated iteratively as before* [see (47), (49), (50)].

Remark. It is clear that this result applies to a smaller rectangle having two opposing vertices on the arc AB, but uniqueness does not hold for a domain contained in ABCD containing only an arc of AB.

CONSEQUENCES

The Riemann formula therefore is true (a and b having first derivatives) provided that the Riemann function exists.

OTHER EXISTENCE THEOREMS

Likewise we can show that it is possible to give values of z on two characteristics $x = x_0$, $y = y_0$ (these values being differentiable and equal at the point x_0, y_0) and obtain a unique solution via successive approximation. This theorem (also due to Picard) proves *the existence* of the Riemann function $u(\xi,\eta,x,y)$.

It would be interesting to apply the Riemann method by replacing the function u by a series not explicitly represented. In this case, it would be more simple to apply the Picard solution directly. We would also take z to be on two curves, one of which is a characteristic etc. On these matters we refer to the work of Picard as cited above.

IV. LINEAR EQUATIONS OF ELLIPTIC TYPE

293. The Laplace Equation. Harmonic functions. Green's formula.

The most simple type of elliptic equation is the Laplace Equation. We shall assume that the variables x,y and the function u are essentially real. We know now (see I,161) that every solution admitting continuous first and second partial derivatives in a simply connected domain in which it is uniform, is the real part of a holomorphic function of $x+iy$ defined up to a constant. Such solutions are known as *harmonic functions.* It follows that *a harmonic function is analytic. If an analytic continuation of the harmonic function* u *is achieved, then the resulting function is again analytic.* This results from the fact that Δu remains as zero (a permanence of the functional equations), as this continuation of u is the real part of the continuation of the holomorphic function whose real part is u. In general, this continuation will provide a *multiform harmonic function.*

A conformal transformation (I, 162) changes a holomorphic function into a holomorphic function and it follows that such a transformation changes a harmonic function to a harmonic function. In particular, a translation, a reflection in the x,y plane and a geometric inversion preserve harmonic functions.

If $u(x,y)$ is harmonic, then its derivative with respect to x or y is again a harmonic function, since it can be seen to satisfy the Laplace equation. The partial derivatives of some order of a harmonic function are harmonic functions. In polar coordinates $r =$, θ ($x = r\cos\theta$, $y = r\sin\theta$), the Laplace equation takes the form:

$$\Delta u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$$

(I,119). From this we deduce that the harmonic functions depend only on r or θ, $C\log r + C'$ and $C''\theta + C'''$, where the C are constants. Those functions are harmonic at a finite distance, except at the origin. Moreover, on setting $z = x+iy$, $i = \sqrt{-1}$ $\log r$ and θ are the real part and the coefficient of i of $\log z$. (The coefficient of i of a holomorphic function is also harmonic; the product of a harmonic function and a constant is harmonic.) On differentiating $\log(x^2+y^2)$ with respect to x, we see that $\cos\theta/r$ is also harmonic (except at the origin).

Corresponding to the general theorems relating to holomorphic functions we have the theorems for harmonic functions which can be proved independently. The proofs in question can be extended to harmonic functions of three variables.

Green's formula

This is analogous to the spatial formula (I, 157). It is deduced from the Riemann formula:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\omega = \int_{\Gamma+} Pdx + Qdy \quad .$$

Let us assume that the functions U and V admit continuous second partial derivatives in a bounded domain D and on its boundary Γ composed of a finite number of rectifiable arcs. Let us apply the above formula by taking

$$Q = V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\,, \qquad P = U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y} \quad .$$

We shall obtain:

$$\iint_D (V\Delta U - U\Delta V)d\omega = \int_{\Gamma^+} V\left(\frac{\partial U}{\partial x}dy - \frac{\partial U}{\partial y}dx\right) - U\left(\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}dx\right) . \qquad (55)$$

As is always the case, the integral Γ is taken in the direct sense on the exterior boundary and in the inverse sense on the interior boundaries if there happen to be any.

Let us assume that the arcs of the curves intersecting Γ admit a tangent at each point (at a point of intersection of two arcs constituting Γ, there is a tangent to the right and a tangent to the left), there is then a normal on which we take the positive sense as that which makes the angle $\pi/2$ with that of the tangent taken in the positive sense. This normal will be the *interior normal* to D at the nonangular points of Γ, if α', β' are the direction cosines, α and β those of the tangent taken along the sense of Γ, then we have $\alpha' = -\beta$, $\beta' = \alpha$ and with arc-length s taken to be positive, we also have $\alpha ds = dx$, $\beta ds = dy$. On the other hand, the derivative of U, for example, *taken on the interior normal* is:

$$-\frac{dU}{dn} = \frac{\partial U}{\partial x}\alpha' + \frac{\partial U}{\partial y}\beta' = -\beta\frac{\partial U}{\partial x} + \alpha\frac{\partial U}{\partial y} \,,$$

such that

$$\frac{\partial U}{\partial x}dy - \frac{\partial U}{\partial y}dx = -\frac{dU}{dn}ds \quad .$$

Formula (55) is thus written as:

$$\iint_D (V\Delta U - U\Delta V)\, dw = \int_\Gamma \left(U\frac{dV}{dn} - V\frac{dU}{dn}\right) ds \quad , \qquad (56)$$

where the integral of the second member no longer has a curvilinear integral

but now an ordinary integral (I, 57), which is to be calculated on each arc of Γ, for *positive* ds.

The expression in (56), completely analogous to the spatial formula (I, 157) is *Green's formula.*

APPLICATIONS

(cf. (I, 157)). I. If U is harmonic in D and on Γ, then the formula (56) for $V = 1$ yields

$$\int_\Gamma \frac{dU}{dn}\, ds = 0 \quad . \tag{57}$$

If this equation holds for a function U, *on every closed curve belonging to a domain* D, then ΔU, if it exists and is continuous, will be zero at every point of D since following (56), we will have:

$$\iint_{D'} \Delta U \, d\omega = 0 \quad ,$$

for any D' in D. *The function* U *is harmonic in* D.

II. *Let us assume that* $U(x,y)$ *is harmonic in a domain* D *and on its boundary* Γ *composed of rectifiable arcs admitting a continuous tangent.* Let x_0, y_0 be a point of D. We describe a circle C with center x_0, y_0 with sufficiently small radius ε such that the interior of C and its circumference belong to D. Let us proceed to apply (56) to the domain D_1 obtained by removing the circle C from D and taking

$$V = \frac{1}{2} \log[(x-x_0)^2 + (y-y_0)^2] \quad ,$$

which happens to be harmonic on D_1 and on its boundary. Since the double integral is zero, we obtain

$$\int_C \left(U \frac{dV}{dn} - V \frac{dU}{dn}\right) ds + \int_\Gamma \left(U \frac{dV}{dn} - V \frac{dU}{dn}\right) ds = 0 \quad . \tag{58}$$

The second integral does not depend on ε, the radius of C. In the first $V = \log \varepsilon$ and the normal derivative is taken alongside of D_1, consequently outside of C; this gives

$$\frac{dV}{dn} = \frac{dV}{d\varepsilon} = \frac{1}{\varepsilon} \quad .$$

On account of equation (57) applied to C, the first integral in (58) reduces to

$$\int_C U \frac{dV}{dn}\, ds = \frac{1}{\varepsilon} \int_C U \, ds \quad .$$

When ε tends to zero, then this expression, in which U is continuous, has a limit $2\pi U(x_0,y_0)$. On letting ε tend to zero, the expression in (58), thus leads to the FUNDAMENTAL RESULT:

$$U(x_0,y_0) = \frac{1}{2\pi}\int_\Gamma \left(V\frac{dU}{dn} - U\frac{dV}{dN}\right)ds \ ,$$

$$V = \log r, \qquad r^2 = (x-x_0)^2 + (y-y_0)^2 \ . \tag{59}$$

This formula equates with the Cauchy formula giving the value of a holomorphic function in terms of its values on a simple closed contour and from which it can be deduced. If W is a harmonic function associated to U, i.e. such that $U+iW$ is a holomorphic function of $z=x+iy$, then following the monogeneity conditions (I, 161),

$$\frac{dU}{dn} = -\frac{dW}{ds} \ .$$

The expression (59) is then written by substituting and integrating by parts $V\frac{dW}{ds}$.

$$U(x_0,y_0) = \frac{1}{2\pi}\int_\Gamma \left(-U\frac{dV}{dn} + W\frac{dV}{ds}\right) ds$$

$$= \frac{1}{2\pi}\int_{\Gamma^+} (UdT + WdV) \ ,$$

where T is associated with V. If we set $f(z)=U+iW$, $z_0 = x_0+iy_0$, then we have $V+iT = \log(z-z_0)$ and the above expression becomes

$$\mathfrak{R}f(z_0) = \mathfrak{R}\frac{-i}{2\pi}\int_{\Gamma^+} (U+iW)d(V+iT)$$

$$= \mathfrak{R}\frac{1}{2i\pi}\int_{\Gamma^+} \frac{f(z)}{z-z_0}\,dz \ .$$

($\mathfrak{R}\alpha$ denotes the real part of α).

THE GAUSS FORMULA

If $U(x,y)$ is harmonic in a circle C of center x_0,y_0 and on its circumference, then (59) does apply. If R is taken to be the radius of C, then on C, we have

$$V = \log R \qquad \text{and} \qquad \frac{dV}{dn} = -\frac{1}{R}$$

(the internal normal to C). On account of (57), we obtain the Gauss formula:

$$U(x_0,y_0) = \frac{1}{2\pi R}\int_C U ds = \frac{1}{2\pi}\int_0^{2\pi} U(x_0 + R\cos\theta,\ y_0 + R\sin\theta)\, d\theta \quad (60)$$

THE MAXIMUM PRINCIPLE

Following the Gauss formula, if $U(x,y)$ is harmonic in C and on C and of M and m arc its maximum and minimum respectively, on the circumference C, then we have

$$m \leqslant U(x_0,y_0) \leqslant M \ .$$

Moreover, in order for $U(x_0,y_0) = M$, it is necessary and sufficient that on every circumference C we have $U(x,y) = M$ (cf. I, 184). If on C we have $U(x,y) = M$, then $U(x,y)$ is constant in C and is equal to M. Effectively, if $P_1(x_1,y_1)$ is an interior point of C, then via an inversion leaving the circumference C fixed, we can transform P_1 to the P_0 of C; U changes to U' harmonic and following the above discussion we have $U(x_1,y_1) = U'(x_0,y_0) = M$. In view of the fact that the analytic continuation of a constant, is a constant, we obtain the theorem of the maximum (cf. I, 184):

If a function U *is harmonic (uniform) in a domain* D *and is non-constant, then the maximum and minimum of its values in some region completely contained within* D*, are attained on the boundary of this region.*

(A region is regarded as a domain along with its boundary.) This theorem also results from the corresponding Cauchy theorem (I, 184). If W is associated to U, then it suffices to apply the Cauchy theorem to

$$e^{U+iW} \quad \text{and} \quad e^{-U-iW} \ .$$

294. The Dirichlet problem. Green's function. The Poisson formula. Harnack's theorem.

The (interior) Dirichlet problem for a bounded domain D whose boundary consists of a finite number of arcs of continuous simple curves, entails finding a harmonic function U in D and taking presribed values on Γ. It is to be understood that when a point P of D tends towards a point Q of Γ, the value $U(P)$ at P must have as its limit the given value at the point Q; we shall call this given value $U(Q)$.

The formula in (59) does not lend itself to a solution of a problem of this kind because of the presence of the term dU/dn. We obtain an expression leading to the solution of the Dirichlet problem, in regard of certain additional hypotheses by replacing the function V which appears by Green's function.

GREEN'S FUNCTION

Consider the given domain D, *its boundary* Γ *and a point* $P_0(x_0,y_0)$ *of* D. *The Green's function for* D *and* P_0 *is a function* $G \equiv G(x,y,x_0,y_0,D)$ *such that*

1° *the function* G *takes the value zero at every point of* Γ;

2° $G + \frac{1}{2}\log[(x-x_0)^2 + (y-y_0)^2]$ *is harmonic in* D.

Such a function, if it exists, is unique by virtue of the maximum principle.

By applying this same principle to the domain obtained by removing from D a small circle centered at P_0 on the circumference of which G is positive, following the second property, we see that G *is positive in* D.

EXISTENCE OF GREEN'S FUNCTION WHEN D IS SIMPLY CONNECTED

In this case we know that we can achieve a conformal representation of D on a circle of radius 1 in such a way that the point P_0 is represented at the center of the circle (I, 215). If we set $z = x + iy$, then a function $Z = f(z)$ providing this representation on $|Z| < 1$ is holomorphic in D and admits the point z_0 ($z_0 = x_0 + iy_0$) as a simple zero. When z tends to the boundary of D, $|Z|$ tends towards 1. The function

$$G(x,y,x_0,y_0,D) = -\mathfrak{R} \log f(z) \tag{61}$$

is harmonic in D, except at the point (x_0,y_0) and is zero on Γ. Moreover, the quotient of $f(z)$ by $z - z_0$ being holomorphic and nonzero in D, means that $\mathfrak{R}\log f(z) - \mathfrak{R}\log(z-z_0)$ is harmonic throughout D, which proves that $G + \log|z-z_0|$ is harmonic in D. Consequently:

The equality in (61) *defines Green's function.*

Remarks. I. We know that all functions achieving the conformal representation in question of D onto the circle $|Z| < 1$ are given by $Z = f(z)e^{i\omega}$ where ω is real. For all these functions, G indeed has the same value.

II. Conversely, if $G(x,y,x_0,y_0,D)$ is known and if we add to $G + \log|z-z_0|$ the product with i of the associated function (defined to within a constant), then we obtain a function $F(z) + \log(z-z_0)$ and

$$Z = e^{-F(z)}$$

gives the desired conformal representation of D onto the circle $|Z| < 1$. For, G being unique and the conformal representation achieved by a function $f(z)$, we necessarily have $\mathfrak{R}F(z) = \mathfrak{R} \log f(z)$.

III. In the conformal representation in question, a point $P_1(x_1,y_1)$ has a corresponding point Z_1. In a conformal representation in which the point

P_1 would have as its image the center of the circle, the point $P_0(x_0,y_0)$ would have as its image a point Z_0. We pass from one of these representations to another by a conformal representation of the circle onto itself. It follows that $|Z_0| = |Z_1|$ and following (61), we obtain

$$G(x_1,y_1,x_0,y_0,D) = G(x_0,y_0,x_1,y_1,D) ,$$

for any (x_0,y_0), (x_1,y_1), thus:

In (61), the points (x,y), (x_0,y_0) play a symmetric part.

THE CASE WHERE THE BOUNDARY Γ OF D IS FORMED BY A FINITE NUMBER OF ANALYTIC ARCS[(1)]

In this case we know that the function $f(z)$ giving the conformal representation is again holomorphic within the interior of these arcs and is extendable beyond them (I, 217). Consequently, the Green function is harmonic within these arcs and is likewise extendable.

THE FORMULA SOLVING THE DIRICHLET PROBLEM

Let us suppose that the Green's function is known for a domain D *whose boundary* Γ *is formed by rectifiable arcs having a continuous tangent and* D *is taken to be in such a way that this function* G *admits continuous first and second partial derivatives on the boundary.* We can apply (56) by taking V to be the function G and by removing the circle C with center (x_0,y_0) and radius ε from D, provided that U, assumed to be harmonic in D, also has the required derivatives on Γ. Proceeding as we did in the proof of formula (59), we will then have

$$\int_C \left(G\frac{dU}{dn} - U\frac{dG}{dn}\right) ds = \int_\Gamma U\frac{dG}{dn}\,ds \quad , \tag{62}$$

since G is zero on Γ. In the neighborhood of the point (x_0,y_0), G is equal to

$$-\frac{1}{2}\log[(x-x_0)^2 + (y-y_0)^2] + H(x,y) \quad ,$$

where $H(x,y)$ is harmonic at this point. Then as we saw above, the first member of (62) tends towards $2\pi U(x_0,y_0)$ in such a way that

$$U(x_0,y_0) = \frac{1}{2\pi}\int_\Gamma U\frac{dG}{dn}\,ds \quad . \tag{63}$$

(1) Here and in what follows we shall say, in short that an arc is analytic when it is a simple analytic arc, without singular points in its interior. The theorem relating to conformal representation (I, 217) is valid for these arcs whose extremities may happen to be singular.

Given this formula, *let us assume that the domain* D *satisfies the above requirements. Let us consider on* Γ *a function* $u(s)$, *continuous, or at the very least, integrable, and consider a priori the integral*

$$U(x_0,y_0) = \frac{1}{2\pi}\int_\Gamma u(s)\,\frac{dG}{dn}\,ds \quad . \tag{64}$$

If this function of (x_0,y_0) *is harmonic in* D *and takes the given values* $u(s)$ *on* Γ, *then* (64) *solves the Dirichlet problem.* As G is zero on Γ and positive in D, dG/dn is positive.

THE CASE OF THE CIRCLE. THE POISSON FORMULA.

If D is a circle of center O and radius R, if A is an interior point of the circle and B the image of A with respect to the circle (A and B are inverses of each other under the inversion which preserves each point of the circumference), we know that for any point P on the circumference, the ratio of PB to PA is constant, and on taking P to be on the segment AB, we see that this ratio is $R/|\vec{OA}|$. The function of the point Q

$$\frac{|\vec{QB}|}{|\vec{QA}|}\;\frac{|\vec{OA}|}{R}$$

is therefore equal to 1 on the circumference, and the Green's function, denoted by $G(Q,A,R)$ is given by

$$G(Q,A,R) = \log\left(\frac{|\vec{QB}|}{|\vec{QA}|}\;\frac{|\vec{OA}|}{R}\right) \quad .$$

Since if we add $\log|\vec{QA}|$ to this function, we obtain a harmonic function in the circle. If we call r,α the polar coordinates of A, ρ,ψ those of Q, then we

$$|\vec{QA}|^2 = \rho^2 + r^2 - 2\rho r\cos(\psi-\alpha)$$

$$|\vec{QB}|^2 = \rho^2 + \frac{R^4}{r^2} - \frac{2R^2\rho}{r}\cos(\psi-\alpha) \quad .$$

and we obtain

$$\frac{dG}{dn} = \left(-\frac{dG}{d\rho}\right)_{\rho=R} = \frac{R^2-r^2}{R^2+r^2-2r\cos(\psi-\alpha)}\,\frac{1}{R} \quad .$$

On replacing $u(s)$ by $u(\psi)$ formula (64) is then written as

$$U(Q) = U(r,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(\psi)\,\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\psi-\alpha)}\,d\psi \quad . \tag{65}$$

This is Poisson's formula. We have seen (I, 208) that *this equation defines a harmonic function in the circle, which takes the assigned values* $u(\psi)$ *on the circumference, provided that* $u(\psi)$ *is a continuous periodic function of period* 2π. The convergence of U towards $u(\psi)$ is uniform.

If $u(\psi)$ has isolated discontinuities and remains integrable, then the result holds true, but the function U only takes the values $u(\psi)$ at the points of continuity. Faton showed that, if $u(\psi)$ is only assumed to be integrable in the Lebesgue sense, then U is harmonic in the circle, and when the point Q tends towards a point P of the circumference along the radius OP, then $U(Q)$ tends towards the given value $u(\psi)$ of u at the point P, with the exception of, at most, a set of measure zero of points of P.

THE CASE OF ARBITRARY DOMAINS

If a *simply connected* domain D is conformally represented on a circle in such a way that there exists a bijective correspondence between the points of the boundary Γ of D and the circumference, then the Dirichlet problem is solvable for D since this is the case for the circle. By virtue of a theorem of Carathéodory which was previously cited (I, 127), *this is the case when the boundary of* D *is a simple Jordan curve.*

A conformal representation of a domain D (whether it is simply connected *or not*) onto a domain D', induces a correspondence of the Green's function of D to that of D': if $G(x,y,x_0,y_0,D)$ is the function relative to D, then that of D' is obtained by replacing x and y by the functions $x = \lambda(x',y')$, $y = \mu(x',y')$ defining the transformation and x_0 and y_0 by $\lambda(x_0',y_0')$, $\mu(x_0',y_0')$. We see at once that the second property of the Green's function is preserved. The Dirichlet problem is solvable for one of the domains when it is so for the other and when the boundaries are in a bijective correspondence, but if (64) happens to be true for one of the domains, then it does not necessarily follow that it is true for the other. *But let us suppose that the boundary* Γ *of* D *is composed of a finite number of analytic arcs and that the boundary* Γ' *of* D' *corresponding bijectively to that of* D, *is also formed by a finite number of analytic arcs.* Then the function defining the correspondence between D and D', $z' = f(z)$, is holomorphic on the boundary of D, except at the points corresponding to the extremities of arcs on one or the other boundary. This is a consequence of the Schwarz reflection principle (I, 216): if z_0 is a point of Γ, z_0' the corresponding point of Γ', both of which are ordinary points, then we can conformally represent a part of D' near to z_0' in such a way that to Γ' there corresponds a part of a line through $z'' = F(z')$. Then $z'' = F(f(z))$ is again holomorphic at the point z_0, hence also $f(z) = F_{-1}(z'')$ (I, 127). Thus $z' = f(z)$ is holomorphic on Γ, except at the points corresponding to vertices. *If* $G(z,z_0,D)$, *the Green's function of* D *relative to*

$z_0 = x_0 + iy_0$, *is again harmonic on the arcs of* Γ, *then the transformed function* $G(z',z_0',D')$ *will again be analytic, hence harmonic on the arcs of* Γ' (except at the points corresponding to the vertices). Under these conditions, *if formula* (64) *applies to* D, *then the transformed formula viz.*

$$2\pi U(x_0',y_0') = \int_{\Gamma'} u(s') \frac{dG(z,z_0',D)}{dn'} ds'$$

will solve the Dirichlet problem in D'.[1]

In particular, *if* D *is simply connected and bounded by a finite number of analytic arcs,* then we know that the conformal representation onto a circle occurs with bijective correspondence of the contours (I, 217). On the other hand, the Green's function of the circle, *given above*, is again harmonic on the circumference. The Green's function of D can be deduced from that of the circle as was remarked. *The formula* (64) *applies under the same conditions as for a circle.*

<u>Harnack's Theorem</u>. This is a proposition analogous to the Weierstrass theorem relating to sequences of holomorphic functions.

Let us consider a sequence of harmonic functions $u_n(Q)$ *in a domain* D, *that is continuous within* D *and on its boundary* Γ, *and let us assume that this sequence converges uniformly on* Γ. *Under these conditions, the sequence converges uniformly in* $D+\Gamma$, *and the limit function* U *is harmonic in* D. *The sequence of derivatives of the same order of the* $u_n(Q)$ *converges uniformly, throughout the domain completely within* D, *towards the derivative of the same order of* U.

If Q belongs to $D+\Gamma$, then the maximum of $|u_n(Q)-u_m(Q)|$ is attained on Γ on account of the maximum principal. The uniform convergence on Γ implies uniform convergence in $D+\Gamma$. To see that the limit function U is harmonic in D, it suffices to show that it is harmonic in every circle C belonging to D as well as on the circumference of C. If Q is inside C, P on the circumference of C, R the radius of C, r the distance of Q from the center and ρ the distance of P to Q, then the Poisson formula gives

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[1] Having applied (64) to D for any $u(s)$, we can see that the integral of dG/dn on Γ is defined and equals 1 following (63).

$$u_m(Q) = \frac{1}{2\pi} \int_C u_m(P) \frac{R^2-r^2}{\rho^2 R} \, ds \quad .$$

Allowing u_m to increase indefinitely, $u_m(P)$ tends uniformly to $U(P)$ and $u_m(Q)$ to $U(Q)$, it follows that

$$U(Q) = \frac{1}{2\pi} \int_C U(P) \frac{R^2-r^2}{\rho^2 R} \, ds \quad , \tag{66}$$

and $U(Q)$ is harmonic in C since the Poisson formula defines a harmonic function. To prove the last part, we note that if x is the abscissa of Q, then the derivative of the last formula with respect to x, gives

$$\frac{\partial U(Q)}{\partial x} = \frac{1}{2\pi} \int_C \frac{\partial}{\partial x} \left(\frac{R^2-r^2}{\rho^2 R} \right) U(P) \, ds$$

(a formula quite similar to the Cauchy formula relating to derivatives of holomorphic functions in terms of the values of the function). We have the same formula for the derivative of $u_m(Q)$ and the uniform convergence of $u_m(P)$ to $U(P)$ implies that of the derivative of $u_m(Q)$ to the derivative of $U(Q)$.

Remark. Following the Poisson formula, if $U(Q)$ is harmonic and positive in the circle C of radius R, then in the above notation, we have

$$\frac{R-r}{R+r} U(Q_0) < U(Q) < \frac{R+r}{R-r} U(Q_0) \qquad r = |\overrightarrow{QQ_0}| \quad , \tag{67}$$

where Q_0 is the center of the circle. Since, in formula (66), ρ has the bounds $R+r$ and $R-r$, and on account of the Gauss formula in (60)

$$\int_C U ds = 2\pi R U(Q_0) \quad .$$

From this inequality we deduce *that a harmonic function on the entire plane at a finite distance which is positive*, is constant. Since we can apply (67) to two any points Q and Q_0 by taking R sufficiently large, whence it follows that $U(Q) = U(Q_0)$. More generally, on applying this result to $U + C'$, where C' is a constant, we see that: *A harmonic function on the entire plane and whose values are bounded to the left or to the right by a fixed number, is a constant.*

This result can be deduced by applying Liouville's theorem to e^f or e^{-f}, where f is a holomorphic function whose real part is U. The inequality (67) also arises from the properties of subbounded functions (I, 216).

295. An alternative procedure due to Schwarz. Special cases and examples.

We shall dwell upon two lemmata.

Lemma I. *If* D *is a domain bounded by a simple Jordan curbe* Γ *and if* $A_1, A_2, \ldots, A_p$ *are arcs of* Γ *without common points, then there exists a unique harmonic function bounded in* D*, taking the value* 1 *on the arcs* A_j *and the value* 0 *on the parts of* Γ *outside of these arcs.*

Effectively, a conformal representation reduces to the case of the circle. The Poisson formula will then provide a solution, which is unique. Since if theere were two solutions, their difference would be a harmonic function bounded in Γ, and taking the value 0 on Γ, except perhaps at the extremities of the arcs A_j. If F is a holomorphic function in D associated to U, then e^F and e^{-F} are bounded in D and their absolute values take the value 1 on Γ, with the exception of the extremities of the A_j. On account of the Lindelöf theorem (I, 242), the absolute values of these functions are at most equal to 1 in D, hence e^F has modulus 1 in D, F is a constant with real part zero by Cauchy's theorem and U is zero

Remarks. I. We considered Carathéodory's theorem on conformal representation. If we wish to restrict our attention to the nature of those results previously verified, then we shall assume here and in what follows that the boundaries of these domains are composed of a finite number of analytic arcs.

II. Some results analogous to Lindelöf's theorem (1908) for harmonic functions were originally proposed by Zaremba (1909).

III. The uniqueness property in Lemma I will not be essential in the application of the lemma to what follows.

Lemma II. *Let* D *be the domain of Lemma I,* $T_1, T_2, \ldots, T_q$ *transversal curves in* D *joining the points of* Γ *outside of the arcs* A_j*. Then the harmonic function defined in Lemma I has a maximum* m *between* 0 *and* 1 *attained on these transversals. If* V *is a harmonic function in* D *taking continuous values on* Γ*, zero outside of the arcs* A_j*, then on the transversals* T_k*, we have* $|V| \leq mK$*, where* K *is the bound of* $|V|$ *on* Γ.

In order to establish the proposition, we can take Γ to be a circumference with center the origin. The function U of Lemma 1 is defined by

$$U(Q) = \frac{1}{2\pi} \sum \int_{A_j} \frac{R^2 - r^2}{R\rho^2}\, ds = 1 - \sum \frac{1}{2\pi} \int_{A_j'} \frac{R^2 - r^2}{R\rho^2}\, ds ,$$

where the A_j' are complementary arcs to the A_j. As $\rho \leq R + r$, we see that, a_j' is the measure of A_j' in radians,

$$U(Q) \leqslant 1 - \sum \frac{a_j'}{2\pi} \frac{R-r}{R+r} \ , \qquad |\overline{OQ}| = r \ ,$$

$U(Q)$ is then less than a number less than 1 for all points Q of the transversals T_k (fig. 86) for which $|\overline{OQ}| < R$. On the other hand, on the arcs of the T_k, the function $U(Q)$ tends to zero when Q approaches the circumference. It is then clear that the maximum m of $U(Q)$ on the T_k is less than 1.

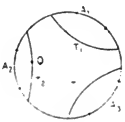

Fig. 86

On the other hand, $V(Q)$ is also given by the Poisson formula and consequently, in the circle

$$KU - V = \frac{1}{2\pi} \int_{\Gamma} [KU(P) - V(P)] \ \frac{R^2-r^2}{R\rho^2} \ ds \geqslant 0 \quad ,$$

since $KU-V$ is positive or zero on all of Γ. It follows that, on the curves T_k, inside of Γ, $V \leqslant KU \leqslant Km$. Likewise, we argue for $KU+V$; this proves the lemma, since $V=0$ on the A_j'.

Remark. The proposition holds true when the T_k have common extremities with the A_j, but with the condition of meeting internally the interior angles of the domain of these points. In this case it can be shown that m is again less than 1.

THE SCHWARZ METHOD

By the Schwarz method we propose to demonstrate that *the Dirichlet problem has a solution when the domain* D *is finitely connected and bounded*

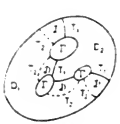

Fig. 87.

by arcs of simple Jordan curves.

We proceed with the proof as follows: To ideas, let us assume that there are three internal boundary curves (fig. 87). On the boundaries Γ we assign the values $u(P)$ of the function to be determined, where $u(P)$ is taken to be continuous on the curves Γ. We consider the transversal T_1 (fig. 87) which split D into two simply connected domains. Let D_1 be one of them, and D_2 the other. In D_1 we consider a new system of transversals T_2 close to the first but not having common points with them and joining the boundaries Γ in the same way as the transversals T_1. The curves Γ, T_1 and T_2 bound four domains (k in the case of k boundary curves Γ) which we shall call $\mathcal{D}$ and which are indicated in fig. 87. If we add to the domain D_2' the domains $\mathcal{D}$ and the curves T_1, we obtain a domain $\mathcal{D}_2$, say, which is simply connected and whose boundary is composed of portions Γ_2 of curves Γ and transversals T_2. The domains D_1 and D_2 have the domains $\mathcal{D}$ in common. We shall let Γ_1 denote the parts of curves Γ which are boundaries of D_1 and P to denote an arbitrary point on the Γ_1, Γ_2, T_1 ... T_2.

On Γ we know the values $u(P)$ of the desired function. To the T_1 let us assign continuous values $u_1(P)$ and consider the values of $u(P)$ at the common points of the Γ_1 and T_1. As D_1 is simply connected, the Dirichlet problem is possible: there exists a function U_1 harmonic in D_1 and taking the values $u(P)$ on the Γ_1 and $u_1(P)$ on T_1. On the T_1 this function takes the value that we shall denote by $\nu_1(P)$ these are equal to the $u(P)$ at the common points of the Γ_2 and T_2. Given that D_2 is simply connected let V_1 be the harmonic function in D_2 which takes the values $u(P)$ on the Γ_2 and $\nu_1(P)$ on T. V_1 takes the values $u_2(P)$ say and at the points common to Γ_1 and T_1 we always have $u_2(P) = u(P)$. Starting from these $u(P_2)$ we can repeat these same two operations: the harmonic function U_2 defined by the $u(P)$ on Γ_1 and the $u_2(P)$ on T_1 takes values $\nu_2(P)$ on T_2 which determine with the $u(P)$ on Γ_2 a harmonic function V_2 in D_2 and so on: U_n takes values $v_n(P)$, on the T_2, that along with $u(P)$ on Γ_2 define a function V_n in D_2; V_n takes values U_{n+1} on T_1 which serve to define U_{n+1}.

We are going to show that the sequence U_n, $n = 1,2,...$ converges uniformly on $\Gamma_1 + T_1$. As a consequence of Harnack's theorem this sequence will converge uniformly in $D_1 + \Gamma_1 + T_1$ to a harmonic function U in D_1 and which takes the values $u(P)$ on Γ_1 and likewise, the V_n will converge uniformly in $D_2 + \Gamma_2 + T_2$ to a harmonic function V in D_2 and which takes on Γ_2 the values $u(P)$. Finally, we shall prove that, in the domains $\mathcal{D}$, $U_n - V_n$ tends to zero, implying $U \equiv V$ in $\mathcal{D}$, and by taking U in D_1 and V in D_2, we will have a function in all of D which will be the desired solutions.

The difference $U_{n+1} - U_n$ is harmonic in D_1 and takes the value 0 on Γ_1 and on T_1 the values $u_{n+1}(P) - u_n(P)$ constitute a continuous function on the boundary of D_1. If K_n is the maximum of $|u_{n+1}(P) - u_n(P)|$ on the boundary and m_1 the quantity appearing in Lemma II relating to D_1 and the transversals T_2, then on the T_2 we have

$$|v_{n+1}(P) - v_n(P)| \leqslant m_1 K_n \quad .$$

Likewise, by applying Lemma II to D_2 and to the transversals T_1, denoting by m_2 the quantity relating to these and by K'_n the maximum on T_2 of $|v_{n+1}(P) - v_n(P)|$, then we shall have

$$K_{n+1} \leqslant m_2 K'_n \quad ,$$

with the above inequality written as

$$K'_n \leqslant m_1 K_n \quad .$$

Consequently,

$$K_{n+1} \leqslant m_1 m_2 K_n \quad , \qquad\qquad n = 1,2,\ldots$$

and by the appropriate multiplication

$$K_n \leqslant (m_1 m_2)^{n-1} K_1 , \qquad K'_n \leqslant m_1 K_n \; , \qquad m_1 < 1 , \;\; m_2 < 1 \quad .$$

It follows that the sequences U_n and V_n converge uniformly on $\Gamma_1 + T_1$ and $\Gamma_2 + T_2$, respectively. Let us now consider a domain $\mathfrak{D}$. The difference $U_n - V_n$ is harmonic there and continuous on the boundary. On segments of Γ its value is zero, on T_1 it is $u_n(P) - u_{n+1}(P)$ and again zero on T_2. Following the maximum principle we shall have in D

$$|U_n - V_n| < K_n \quad ,$$

a quantity which tends towards zero. The existence theorem for the solution of the Dirichlet problem is established. As $u(P)$ is continuous, the Poisson formula applied to the images of D_1 or D_2 proves that $U(M)$ tends uniformly towards $U(P)$ if $M \to P$.

EXISTENCE OF THE GREEN'S FUNCTION. FOR MULTIPLY CONNECTED DOMAINS.

If D is a finitely connected domain bounded by arcs of simple curves, then we obtain the Green's function $G(P,Q,D)$ relative to an interior point Q, where P is an arbitrary interior point, by determining the complementary function

$$H(P,Q,D) = G(P,Q,D) + \log|\overline{PQ}|$$

which is harmonic in D and takes the continuous values $\log|\overline{PQ}|$ on the boundary. This function exists on account of the Schwarz's theorem, hence also G exists.

THE CASE OF GIVEN ANALYTIC VALUES

Let us assume that the boundary curves of D are formed by a finite number of analytic arcs and that within the interior of each of these arcs, the given values $u(P)$ are the values of an analytic function of the parameter defining this arc [this will be the case if $u(P)$ is an analytic function with coordinates x,y, of the point P]. Under these conditions, the values taken on the boundary Γ by the solution U of the Dirichlet problem, are analytic.

Let us consider the neighborhood of an ordinary point P_0 of Γ, i.e., the part D' of D belonging to a circle of center P_0 and with small radius. This radius can be assumed to be so small to permit D' to be simply connected. Let then W be a function associated with U in D', $U+iW$ is a holomorphic function of $z=x+iy$ in D'. By means of a conformal representation, we can revert to the case where the part of Γ, which is a part of the boundary of D', is a segment of the axis Ox; $U+iW$ will be holomorphic for $y>0$ for example and its real part U will take on the segment α,β of Ox the values of an analytic function $\psi(x)$. If we replace x by $z=x+iy$ in $\psi(x)$, we obtain a function $\psi(z)$ holomorphic about z_0 assigned to P_0. The real part $U+iW-\psi(z)$ tends uniformly towards zero if $y>0$ tends towards zero and x towards a neighboring value of x_0. On account of symmetry, this function is extendable beyond z_0, consequently $U+iW$ is extendable and U is indeed harmonic at the point P_0. It follows that:

When the boundary of D *is formed by analytic arcs on which analytic values are prescribed, then the solution of the Dirichlet problem is again harmonic on these arcs and, to extent, beyond them.*

In particular, *when the boundary* Γ *of* D *is formed by analytic arcs, the Green's function* $G(x,y,x_0,y_0)$ *is again harmonic on these arcs and to small extent beyond them;* $\partial G/dn$ *exists and is analytic on these arcs.* Formula (63) then applies to the solutions of the Dirichlet problem which are continuous on Γ along with their two first order partial derivatives. Formula (64) applies to the conditions which have been stated, in particular, when each of the closed curves bounding D is analytic without singular points, and if $u(s)$ is analytic on each of them.

THE DIRICHLET PROBLEM AND THE CAUCHY PROBLEM

Let us take for example a simple closed analytic curve Γ (circle, ellipse, etc.) and consider on Γ some analytic values $u(P)$. There exists a function $u(Q)$ harmonic in the interior of Γ and taking these values on

Γ. Let us also consider a second series of analytic values $u_1(P)$. Following the Cauchy theorem, there exists in the neighborhood of a point P_0 of Γ a unique solution of the equation $\Delta U = 0$, $U_1(Q)$ say, which takes the values $u(P)$ on Γ and whose derivative with respect to x takes the values $u_1(P)$. This solution $U_1(Q)$ can be extended along Γ and will take the values $u(P)$; its derivative with respect to x will take the values $u_1(P)$. But, with the exception of those values $u_1(P)$ which are precisely those corresponding to the solution $U(Q)$ of the Dirichlet problem, $U_1(Q)$ will not be harmonic throughout the interior of Γ. The Dirichlet problem exists and is harmonic throughout the interior of Γ, whilst that of the Cauchy problem only exists in a neighborhood of Γ which depends on the initial conditions.

AN EXAMPLE OF DOUBLY CONNECTED DOMAINS

It can be shown that a finite doubly-connected domain, bounded by two simple curves, can be conformally representable onto a circular ring for which the ratio of the radii is suitably chosen (see Julia: *Leçons sur la representation conforme des aires multiplement connexes*, Paris, 1934). If the boundaries of the domain D consist of analytic arcs, then on account of the discussion in no. 293, it will suffice to show that formula (64) holds in a circular ring, in order to show that it is tenable in D.

THE GREEN'S FUNCTION FOR AN ANNULUS

This is obtained by generalizing the procedure for a circle.
Consider the annulus

$$R < |z| < R\sqrt{s} = R' , \qquad s' \text{ real} > 1 .$$

If α is an interior point of the ring and $\alpha' = R'^2/\bar{\alpha}$ its image with respect to the circumference $|z| = R'$, then we have $R' < |\alpha'| < Rs$. Let us consider the loxodromic function $S(z)$ corresponding to the real multiplier s (I, 235), we can consider

$$N(z) \equiv \frac{S(\frac{z}{\alpha'})}{S(\frac{z}{\alpha})} \equiv -\frac{z}{\alpha'} \prod_0^\infty \frac{1-\frac{\alpha'}{zs^n}}{1-\frac{z}{\alpha s^n}} \prod_0^\infty \frac{1-\frac{z}{\alpha' s^n}}{1-\frac{\alpha}{zs^n}}$$

$$\equiv \prod_0^\infty \frac{1-\frac{\alpha'}{zs^{n+1}}}{1-\frac{z}{\alpha s^n}} \prod_0^\infty \frac{1-\frac{z}{\alpha' s^{n-1}}}{1-\frac{\alpha}{zs^n}} .$$

For $|z| = R'$, each ratio of the first line has a constant modulus, for $|z| = R$, each ratio of the second line has a constant modulus. Consequently

$N(z)$ is constant for $|z| = R$. On noting that $S(z)$ is real in this case, hence having the same modulus as $S(\bar{z})$, and applying the properties of this function (I, 235), we obtain, for the value of $|N(z)|$ on $|z| = R$,

$$|S(R/\alpha')/S(R/\alpha)| = |S(\bar{\alpha}/Rs)/S(R/\alpha)|$$

$$= |RS(\alpha/R)/S(R/\alpha)\alpha| = 1 \quad .$$

For $|z| = R'$, the modulus of $N(z)$ is

$$|S(R'/\alpha')/S(R'/\alpha)| = |S(\alpha/R')/S(R'/\alpha)| = |\alpha|/R' \quad .$$

The Green's function of the annulus is the real part of

$$K(z) = \log S\left(\frac{z}{\alpha'}\right) - \log S\left(\frac{z}{\alpha}\right) + 2\,\frac{\log R' - \log|\alpha|}{\log s}\,\log\frac{z}{R} \quad ,$$

where the third term is determined by the condition $\Re K(z) = 0$ for $|z| = R$ and $|z| = R'$. This formula can be expressed by introducing the elliptic Weierstrass function σ or Jacobi function $H(u)$ (I, 237); this enables us to recover some results due to Villat. Let us calculate dG/dn. Since G is the real part of $K(z)$, analytic except at the point α, we have for $|z| = R$,

$$\frac{dG}{dn} = \frac{\partial G}{\partial x}\frac{x}{R} + \frac{\partial G}{\partial y}\frac{y}{R} = \Re\left(K'(z)\,\frac{z}{R}\right) \;;$$

for $|z| = R'$; it is necessary to change the denominator in the third member from R to $-R'$. The normal derivative is expressed in terms of $K'(z)$, i.e. in terms of the loxodromic function $\chi(z)$ (I, 236). For $|z| = R$,

$$\frac{dG}{dn} = \frac{1}{R}\,\Re\left[\chi\left(\frac{z}{\alpha'}\right) - \chi\left(\frac{z}{\alpha}\right) + 2\,\frac{\log R' - \log\alpha}{\log s}\right] \quad . \tag{68}$$

Since χ is real, we can replace z/α' by the conjugate number $\bar{z}\alpha/R'^2$ in (68). The first two terms in parenthesis become

$$\chi\left(\frac{\bar{z}\alpha}{R^2 s}\right) - \chi\left(\frac{z}{\alpha}\right)$$

and define a function of α holomorphic in the annulus and continuous with respect to α and z, where z is the circumference $|z| = R$. On account of the properties of integration of holomorphic functions depending on one parameter (I, 240) if $u_1(\psi)$ is the given value at the point $Re^{i\psi}$, then the integral

$$\int u_1(\psi)\,\frac{dG}{dn}\,R\,d\psi$$

is the real part of a holomorphic function of α plus a linear term in $\log\alpha$ and this happens to be a harmonic function of the coordinates of α.

We have an analogous result for $|z| = R'$ and explicitly we can write

$$U(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u_1(\psi)\mathcal{R}\left[-\chi\left(\frac{Re^{i\psi}}{\alpha}\right)+\chi\left(\frac{ae^{-i\psi}}{Rs}\right) - 2\,\frac{\log\alpha - \log R'}{\log s}\right]d\psi$$

$$+\frac{1}{2\pi}\int_0^{2\pi} u_2(\psi)\mathcal{R}\left[\chi\left(\frac{R'e^{i\psi}}{\alpha}\right)-\chi\left(\frac{\alpha e^{-i\psi}}{R'}\right)+2\,\frac{\log\alpha - \log R'}{\log s}\right]d\psi\ , \tag{69}$$

where $u_2(\psi)$ is the given function of $|z| = R'$. $U(\alpha)$ is harmonic in the annulus. We assume that u_1 and u_2 are continuous of period 2π. It remains to show that when α tends towards a boundary point, $U(\alpha)$ tends towards the corresponding value of $u_1(\psi)$ or $u_2(\psi)$. We proceed as in the case of the Poisson formula (I, 208).

If in (69) we replace $u_1(\psi)$ and $u_2(\psi)$ by the same constant α, then the formula (which is now in fact (63)) is applied to give $U(\alpha) = a$. In order to show that $U(\alpha)$ tends towards $u_1(\psi_0)$ for example, when α tends to the point $Re^{i\psi_0}$ we can then subtract $u_1(\psi_0)$ from $U(\alpha)$, $u_1(\psi)$, $u_2(\psi)$. Moreover, we can assume that $\psi_0 = 0$ by replacing ψ and α by $\psi-\psi_0$ and $\alpha e\text{-}i\psi$. We have reduced matters to the case where $u_1(0) = 0$ and where α tends towards R, and we need to show that $U(\alpha)$ tends to 0. Let us decompose the first integral into two parts: one taken from $-\eta$ to $+\eta$ and the other from η to $2\pi-\eta$, where η is small, positive. We have $|u_1(\psi)| < \varepsilon$ given when $|\psi| < \eta$. The integral taken from $-\eta$ to η is less in absolute value to that obtained by replacing $u_1(\psi)$ by ε, (dG/dn is positive) and, *a fortiori* by replacing in the second member of (69) $u_1(\psi)$ and $u_2(\psi)$ by ε, which yields ε for any α. On the other hand, the factors of $u_1(\psi)d\psi$ and $u_2(\psi)d\psi$ tend uniformly towards zero when α tends to R when we also assume $|\psi| \geqslant \eta$ in the first integral. For they are contained in α and ψ, since

$$\chi(z) = \sum_0^\infty \frac{z}{z-S^n} + \sum_1^\infty \frac{1}{S^n z-1}\quad ,$$

and since (I, 236), $\chi(sz) \equiv \chi(z) + 1$, we have for $\alpha = R$

$$-\chi(e^{i\psi}) + \chi\left(\frac{e^{i\psi}}{s}\right) = -1,\qquad \mathcal{R}\left[-\chi(e^{i\psi}) + \chi\left(\frac{e^{-i\psi}}{s}\right)\right] = -1$$

$$\chi(\sqrt{s}\,e^{i\psi}) - \chi\left(\frac{e^{i\psi}}{\sqrt{s}}\right) = 1,\qquad \mathcal{R}\left[\chi(\sqrt{s}\,e^{i\psi}) - \chi\left(\frac{e^{-i\psi}}{\sqrt{s}}\right)\right] = 1\ .$$

Finally $2(\log R - \log R') = -\log s$. The proposition follows and the convergence of $U(\alpha)$ towards the values given on the boundary is similarly uniform. Thus:

Formula (69) solves the Dirichlet problem for the annulus provided that the given quantities $u_1(\psi)$ *and* $u_2(\psi)$ *are continuous.* The convergence of $U(\alpha)$ towards these values, is uniform.

Remark. As in the case of the Poisson formula (I, 208), we note that the terms in χ in formula (69) are expendable in a Laurent series in α in the annulus. Consequently, if we set $\alpha = re^{i\theta}$ we can write (69) in the form

$$U(\alpha) = 2\,\frac{\log r}{\log s}\,(A_0'-A_0) + \sum_0^\infty [A_m(a_m r^m + a_m' r^{-m})\cos m\theta + B_m(b_m r^m + b_m' r^{-m})\sin m\theta]$$

$$+ \sum^\infty [A_m'(c_m r^m + c_m' r^{-m})\cos m\theta + B_m'(d_m r^m + d_m' r^{-m})\sin m\theta] \quad , \tag{70}$$

where the A_m, B_m are the Fourier coefficients of $u_1(\psi)$ and the A_m', B_m' those of $u_2(\psi)$ and the a_m, a_m', b_m, b_m', c_m, c_m', d_m, d_m' those of the constants which only depend on R and R'. We can determine these constants directly by considering the case where the second member again converges uniformly for $r = R$ and $r = R'$. We obtain the following:

$$a_0 + a_0' = 2\,\frac{\log R'}{\log s} \quad , \qquad c_0 + c_0' = -\,\frac{2\log R}{\log s} \quad ,$$

$$a_m = b_m = \frac{R^m}{\delta m} \quad , \qquad a_m' = b_m' = -R'^{2m} a_m \quad ,$$

$$c_m = d_m = -\,\frac{R'^m}{\delta m} \quad , \qquad c_m' = d_m' = -R^{2m} c_m \quad ,$$

$$\delta m = R^{2m} - R'^{2m} \quad .$$

296. A linear equation with constant coefficients.

Let us first of all consider an equation of the form

$$\Delta u = f(x,y) \; .$$

We are looking for a continuous solution along with its two first order derivatives in a bounded domain D and *taking the given values* $u(P)$ *on the boundary* Γ. We assume that $f(x,y)$ continuous in $D+\Gamma$ admits continuous first derivatives in D. The equation cannot admit two solutions u_1 and u_2, since $u_1 - u_2$ would then be harmonic in D and zero on Γ, hence

zero. If u_1 is a solution of (71), continuous along with its two first order derivatives in D and if we set $u = u_1 + v$, then u will be a harmonic function which will take the values $u(P) - u_1(P)$ on Γ and will be a solution of the Dirichlet problem. For example, *the equation* $\Delta u = d$, *where* d *is a constant, admits* $d/4(x^2 + y^2)$ as a solution and *to find the solution taking the prescribed values on* Γ, *amounts to the Dirichlet problem.* We also see that in the problem pertinent to (71), we can assume $u(P) \equiv 0$ on Γ; we can reduce matters accordingly by subtracting the harmonic function taking the values $u(P)$ on Γ from the desired solution.

If U is a solution of (71) zero on Γ and if $V = G(P,Q,D)$ is the Green's function relative to the domain D and the point Q of D, then we can apply Green's formula (56) to the domain D' obtained by removing a small circle of center Q and radius ε from D, provided that the required conditions concerning U and $G(P,Q,D)$ are satisfied on Γ. We then proceed as we did to establish formula (63), but here, U is zero on Γ whilst ΔU is nonzero but equal to $f(x,y)$ in D. By letting ε tend to zero, we obtain

$$U(Q) = -\frac{1}{2\pi} \iint_D f(P)G(P,Q,D) \, d\omega \, , \tag{72}$$

where P is a varying point in D. *Such is the formula which will provide the solution.* Now, by applying Green's formula (56) to $G(P,Q,D)$ and to $G(P,Q',D)$ in the domain with the small circles centered at Q and Q' removed, and allowing their radii to tend to zero, we see that in the general case of multiply-connected domains (but with boundaries satisfying the conditions in order that (63) applies to these functions G), as in the case of the simply connected domains, we have $G(Q,Q',D) = G(Q',Q,D)$ where Q and Q' are distinct in D. In the second member of (72) we can put $G(Q,P,D)$ which is the sum of a harmonic function $H(P,Q)$ of the point Q and of $-\log|\overrightarrow{PQ}|$. The double integral of $F(P)H(Q,P)$ defines a harmonic function, and

$$\lambda(Q) = \frac{1}{2\pi} \iint_D f(P)\log|\overrightarrow{PQ}| \, d\omega \tag{73}$$

is known as a logarithmic potential. As $\log|\overrightarrow{PQ}|$ is harmonic in Q except at the point Q, then we can replace D by a small circle containing Q which omits a harmonic function of Q. The direct calculation of $\Delta\lambda(Q)$ when D is a small circle containing Q shows that this quantity is equal to $f(Q)$. The function (72) therefore satisfies equation (71) by means of the hypotheses on Γ. If K is the maximum of $|f(P)|$ in D, we have

$$|U(Q)| \leq \frac{K}{2\pi} \iint_D G(P,Q,D) \, d\omega \quad . \tag{74}$$

Now formula (72) applies when $f(P) = 1$ provided that $G(P,Q,D)$ satisfies the contour conditions and consequently, the second member of (74) tends to zero when Q tends to the contour. Equation (72) provides the solution. In particular: *If* Γ *consists of simple closed analytic curves without singular points, then* (72) *yields the solution of equation* (71) *which takes the value* 0 *on* Γ.

Remark. A conformal representation of D transforms equation (71) into an analogous equation, since $(dx^2 + dy^2)\Delta u$ is invariant. We deduce that formula (72) again holds when Γ consists of a finite member of analytic arcs with bounded curvature.

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation such as

$$\alpha \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial^2 u}{\partial x \partial y} + \gamma \frac{\partial^2 u}{\partial y^2} + \delta \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \varepsilon u = 0 \ , \qquad \beta^2 - \alpha\gamma < 0 \ ,$$

where $\alpha, \ldots, \varepsilon$ are constants, can be simplified on consideration of the characteristics. Matters are reduced to the case $\alpha = \gamma = 1$, $\beta = 0$, then changing the unknown function, as in the hyperbolic case (no. 291). We arrive at the canonical form

$$\Delta u = cu \quad ,$$

where c is a constant. On seeking a zero solution on the boundary Γ of a domain D, admitting continuous first and second partial derivatives in D, we see that formula (72) gives rise to a definition of this solution as a function $U(Q)$ satisfying the equation

$$U(Q) \ = \ - \frac{c}{2\pi} \iint_D U(P) G(P,Q,D) \, d\omega \quad .$$

Such an equation is *an integral equation* with fixed limits, linear (in U) and homogeneous.

Equations of this type were originally studied and solved by Fredholm in 1903. They strike upon a vast number of problems. On this topic we refer the reader to the works of Fréchet and Heywood (*L'equation de Fredholm*), Volterra (*Leçons sur les equations integrales*, 1913), Hilbert (*Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, 1924), Volterra and Pérès (*Théorie des fonctionelles*, vol. I, 1936) and to vol. III of Goursat's *Cours d'analyse*.

V. EQUATIONS OF PARABOLIC TYPE

297. The heat equation

We shall restrict matters to considering the simplest form of a parabolic equation whose solution is not absolutely trivial. This is Fourier's equation which arises in thermodynamics

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \quad . \tag{75}$$

The characteristics are defined by $y = \text{const.}$, x and y are assumed to be real.

ANALYTIC SOLUTIONS

The theorem of Cauchy and Kowalewska (no. 284) applies. If we are given, for example, $u = \phi(y)$, $\partial u/\partial x = \psi(y)$, where ϕ and ψ are analytic on a segment (a,b), then there exists a unique analytic solution which for $x = x_0$, takes the value $\phi(y)$, its derivative with respect to x being $\psi(y)$. In order to determine this solution, it is permissible to write for $x - x_0$ and $y - y_0$ sufficiently small $(a \leqslant y_0 \leqslant b)$,

$$u = \phi(y) + (x - x_0)\psi(y) + \sum_2^\infty \left(\frac{x-x_0}{n!}\right)^n \theta_n(y) \ , \tag{76}$$

where the $\theta_n(y)$ are power series, which on account of the existence theorem, are such that the double series so defined, is convergent. On differentiating u with respect to x twice and with respect to y and stating that (75) is true everywhere, then we see that the coefficients of the various powers of $x - x_0$ become zero; we have

$$\theta_{n+2}(y) = \theta_n'(y) \ , \qquad n = 2,\ldots$$

$$\theta_2(y) = \phi'(y) \quad , \qquad \theta_3(y) = \psi'(y) \quad .$$

Consequently, we have under the indicated conditions

$$u = \phi(y) + (x - x_0)\psi(y) + \cdots + \frac{(x-x_0)^{2n}}{(2n)!} \phi^{(n)}(y)$$
$$+ \frac{(x-x_0)^{2n+1}}{(2n+1)!} \psi^{(n)}(y) + \ldots \quad . \tag{77}$$

But, observing the properties of analytic functions (I, 78, 186), we can find two numbers M and k such that, throughout the segment (a,b),

$$|\phi^{(n)}(y)| < Mk^n n!, \qquad |\psi^{(n)}(y)| < Mk^n n! \qquad n = 0,1,2,\ldots \tag{78}$$

These inequalities show that the expression (77) *is an entire function of* $x - x_0$ *for any* y *on* (a,b). *The solution provided by Cauchy's theorem remains analytic throughout the strip* $a \leq y \leq b$. If we assign a particular value y_0 to y, u becomes an entire function of $(x - x_0)$

$$u = \sum_0^\infty c_n (x - x_0)^n \tag{79}$$

whose coefficients are determined by the values ϕ and ψ and their derivatives at the point y_0:

$$c_{2n} = \frac{\phi^{(n)}(y_0)}{(2n)!}, \qquad c_{2n+1} = \frac{\psi^{(n)}(y_0)}{(2n+1)!}, \qquad n = 0,1,2,\ldots . \tag{80}$$

Following the inequalities (78) and Stirling's formula (I, 81), we have

$$\overline{\lim_{u=\infty}} \quad n\sqrt{|c_n|} \ \sqrt{n} \leq \sqrt{\frac{ke}{4}} \quad , \tag{81}$$

or,

$$\overline{\lim_{u=\infty}} \quad n\sqrt{|c_n|} \quad 2n\sqrt{n!} \leq \sqrt{\frac{k}{4}} \quad . \tag{82}$$

Conversely, if the coefficients of (79) satisfy the condition (81), we can put these under the form (80) and $\phi^{(n)}(y_0)$ and $\psi^{(n)}(y_0)$ will satisfy the inequalities (78) with the condition of replacing k by a value k' greater than k. There will exist functions $\phi(y)$ and $\psi(y)$ admitting these numbers $\phi^{(n)}(y_0)$ and $\psi^{(n)}(y_0)$ for derivatives of order n $(n = 0,1,\ldots)$ at the point y_0, expendable as a Taylor series for $|y - y_0| < 1/k$. From this arises the following:

Every entire function (79) *whose coefficients satisfy a condition of the form* (81) *or* (82), *defines a solution of equation* (75) *which is analytic in the strip* $|y - y_0|$ $k < 1$ *and which takes for* y_0 *the values of this function* (y_0 arbitrary).

<u>Remarks</u>. I. We see that the single assignment of u on a characteristic defines a solution, but u will be subjected to some conditions.

II. The condition (82) implies that the entire function (79) is of order at most two (I, 211) and that when it is of order two, it is of a determinable type (I, 243). These are necessary and sufficient conditions.

III. If we look for, *a priori*, a solution in the form of (76), with ϕ and ψ infinitely differentiable, we will again discover the form (77) but with convergence taken into account. This convergence is guaranteed under far less restrictive conditions than (78). The functions ϕ and ψ could be taken to be quasi analytic (I, 80).

298. The Fourier solution

Equation (75), by virtue of its linearity admits solutions of the form

$$u = e^{ax+by} ,$$

where a and b are constants. We see at once that a and b are related by $b = a^2$. In particular, we can take $a = i\alpha$, α *real* and $i^2 + 1 = 0$ which shows that

$$e^{-\alpha^2 y}\cos \alpha x , \qquad e^{-\alpha^2 y}\sin \alpha x ,$$

are solutions. It follows that

$$u_1 = \int_{-\infty}^{+\infty} e^{-\alpha^2 y + i\alpha x}\, d\alpha , \qquad y > 0$$

is also a solution; this can be seen once we have calculated the derivatives of this function u_1 of x and y. Now this integral is calculated by decomposing the exponent into squares

$$-\alpha^2 y + i\alpha x = -y\left(\alpha - \frac{ix}{2y}\right)^2 - \frac{x^2}{4y} ,$$

which yields

$$u_1 = e^{-x^2/4y} \int_{-\infty}^{+\infty} e^{-yv^2}\, d\alpha , \qquad v = \alpha - \frac{ix}{2y} .$$

If we consider this last integral in the plane of the complex variable $\alpha = \alpha' + i\alpha''$, it is equal to

$$\int e^{-y\alpha^2}\, d\alpha \tag{83}$$

evaluated on the line Δ with equation

$$\alpha'' = -\frac{x}{2y} , \qquad \text{from } -\infty \text{ to } +\infty \text{ (fig. 88).}$$

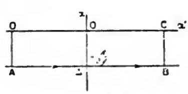

Fig. 88.

The integral (83) evaluated over the contour ABCD is zero, since the function to be integrated is holomorphic in this rectangle. If B is elongated indefinitely to the right, then the integral (83) taken on BC, whose absolute value is less than

$$\left|\frac{x}{2y}\right| e^{-y\lambda} , \qquad \lambda = \overline{OC}^2 - \frac{x^2}{4y^2} ,$$

tends to zero. Likewise, the integral taken on AD tends to zero when D is elongated indefinitely to the left. It follows that the integral (83) evaluated on Δ is equal to this integral evaluated on the real axis:

$$\int_{-\infty}^{+\infty} e^{-y\alpha^2} \, d\alpha = \frac{1}{\sqrt{y}} \int_{-\infty}^{+\infty} e^{-t^2} \, dt = \sqrt{\frac{\pi}{y}} .$$

Consequently,

$$\frac{u_1}{\sqrt{\pi}} = \frac{e^{-x^2/4y}}{\sqrt{y}}$$

is a solution of equation (75), and likewise

$$\frac{e^{-(x-t)^2/4y}}{\sqrt{y}} , \qquad y > 0 ,$$

is a solution for any t. Again it follows that if $f(t)$ is a continuous function for any t and bounded, then the function

$$U = \frac{1}{2\sqrt{\pi}\sqrt{y}} \int_{-\infty}^{+\infty} f(t) e^{-(x-t)^2/4y} \, dt , \qquad y > 0 , \tag{84}$$

is also a solution. We can see this by means of a direct calculation. Clearly, we can replace y by $y - y_0$, $y > y_0$, and broaden the hypotheses on $f(t)$. In particular, it suffices to take $f(t)$ integrable and to have

$$|f(t)| < k' e^{kt^2} , \tag{85}$$

where k and k' are given positive numbers in order that the function U defined by (84) is a solution for any x and providing that y is sufficiently small and positive.

Formula (84) lends itself to a solution of the following heat problem: *to find an integral of equation* (75) *which takes the prescribed continuous values on a characteristic* $y = y_0$. We can, in fact, assume that the characteristic is $y = 0$. *We assume that on the line* $y = 0$ *we are given the values* $u = f(x)$, *where* $f(x)$ *satisfies condition* (85) *and we are going to show that*

when the point $M(x,y)$ *tends towards a point* $(t',0)$ *of the axis* $y=0$, *the function* $U(x,y)$ *defined by* (84) *tends to* $f(t')$. The proof is the same as that for the Poisson formula (I, 208). For $f(t)=1$, the formula (84) yields $U=1$. We therefore have

$$U(x,y) - f(t') = \int_{-\infty}^{+\infty} [f(t) - f(t')]K(t,x,y)\, dt \tag{86}$$

with

$$K(t,x,y) \equiv \frac{e^{-(x-t)^2/4y}}{2\sqrt{\pi y}}, \qquad \int_{-\infty}^{+\infty} K(t,x,y)\, dt \equiv 1 \quad .$$

Given that ε is positive and $f(t)$ is continuous at the point t', we can find η such that $|f(t) - f(t')| < \varepsilon$ if $|t-t'| < \eta$. We can decompose the integral of the second member of (86) into three integrals: one of them I, say, taken from $t'-\eta$ to $t'+\eta$, another I'' from $-\infty$ to $t'-\eta$ and the third I''' from $t'+\eta$ to $+\infty$. For any $x, y>0$, we have

$$|I| < \varepsilon \int_{t'-\eta}^{t'+\eta} K(t,x,y)\, dt < \varepsilon \int_{-\infty}^{\infty} K(t,x,y)\, dt = \varepsilon \quad .$$

We shall proceed to show that if x and y are sufficiently near to t' and 0 respectively, then I' and I'' are as small as is desired which will complete the proof. It is sufficient to consider I'', for example. On account of (85), for m fixed, we have

$$|f(t) - f(t')| < me^{kt^2} \quad ,$$

hence

$$2\sqrt{\pi y}\ |I''| < m \int_{t'+\infty}^{\infty} e^{v}\, dt \quad , \qquad v = kt^2 - \frac{(t-x)^2}{4y} \quad .$$

Throughout the interval of integration, we have

$$v < kt^2 - \frac{\left(t - t' - \frac{\eta}{2}\right)^2}{4y} < -\frac{\left(t - t' - \frac{\eta}{2}\right)^2}{16y}$$

provided that $|x-t'| < \eta/2$ and y is sufficiently small: $y < y_1$. By replacing v by this bound and setting $t = t' + \eta/2 + 4\sqrt{y}\, T$, we obtain

$$2\sqrt{\pi y}\ |I''| < m \int_{T'}^{\infty} e^{-T^2}\, 4\sqrt{y}\ dT, \qquad T' = \frac{\eta}{8\sqrt{y}} \quad ,$$

hence

$$|I''| < \frac{2m}{\sqrt{\pi}} \int_{T'}^{\infty} e^{-T^2} \, dT ,$$

and the second member is as small as is desired providing that T' is sufficiently large, in other words, y sufficiently small. The proposition is thus established.

THE ANALYTICITY OF THE SOLUTION

The solution defined by (84) is an analytic function of the real variables x and y, for $y > 0$. We see this by extending the previous considerations for the one variable case to functions of two variables. If A is taken to be positive and finite, then the integral

$$\int_{-A}^{A} f(t) e^{-(x-t)^2/4y} \, dt \tag{87}$$

defines an analytic function of x and y *assumed to be complex*, if the real part of y is positive (it could also be taken to be negative). To prove this, we can show that it is possible to differentiate with respect to x and y which is clearly the case, and apply the result of no. 51, or even consider the exponential which evolves out of the power series in $x - x_0$ and $y - y_0$ about x_0 and y_0 ($\Re y_0 > 0$). This series is majorized by

$$e^W , \qquad W = \frac{(|x-x_0|+|x_0|+|t|)^2}{4(|y_0|-|y-y_0|)} , \qquad |y - y_0| < |y_0|$$

and since $|t| \leqslant A$, there is uniform convergence. As $|f(t)|$ is bounded, the integral (87) is taken across a uniformly convergent power series and we can integrate term by term.

If we assume $|y-y_0| < \alpha|y_0|$, $2\alpha < 1$, and if x belongs to a bounded domain, then the integral (87) converges uniformly when A tends to infinity. By taking A to be a sequence of integers, we obtain a sequence of holomorphic functions of x and y which converges uniformly. The theorem of Weierstrass relating to sequences of holomorphic functions (I, 186) extends to the case of functions of two variables by applying the Poincaré double integral (no. 51). From this it follows that the function U defined by (84) is an analytic function of x and y for y positive, this function being, moreover, an entire function of $x - x_0$ for any x_0.

The solutions provided by the (84) are therefore less general than those defined by (77), in which, as we observed in no. 296 (Remark III), we can take $\phi(y)$ and $\psi(y)$ to be nonanalytic functions.

299. Uniqueness questions. The general properties of the solutions.

We can define an adjoint equation to equation (75) as we did in no. 290 for the hyperbolic case. If we take $E(u)$ to denote the first member of equation (75), then the adjoint equation is

$$F(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$$

and we have

$$uE(v) - vF(u) \equiv \frac{\partial}{\partial x}\left[u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}\right] - \frac{\partial (uv)}{\partial y} .$$

If u and v have the necessary partial derivatives in a domain Δ and on its boundary Γ, then the Riemann formula shows that

$$\iint_{\Delta} [uE(v) - vF(u)]\, d\omega = \int_{\Gamma^+} uv\, dx + \left[u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}\right] dy . \tag{88}$$

If we take $u \equiv 1$ and $v \equiv U^2$ in this formula, where U is a solution of $E(U) = 0$, then we have

$$E(U^2) \equiv 2UE(U) + 2\left(\frac{\partial U}{\partial x}\right)^2 \equiv 2\left(\frac{\partial U}{\partial x}\right)^2$$

and

$$2 \iint_{\Delta} \left(\frac{\partial U}{\partial x}\right)^2 d\omega = \int_{\Gamma^+} U^2\, dx + 2U \frac{\partial U}{\partial x}\, dy . \tag{89}$$

As a consequence of this relationship, *there can only exist one solution* U *admitting continuous first derivatives and taking continuous values prescribed on a segment of a characteristic and on two continuous curves* AC, BD *intersected at a single point by the characteristics* $y = \text{const.}$ *and located as in* fig. 89.

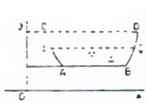

Fig. 89.

In order to establish this, it suffices to show that if a solution U is zero on the contour CABD, it is zero at every point M of a segment PQ for any $y = \text{const.}$ intersecting AC and BD. If we apply formula (89) to the domain Δ bounded by the contour ABQP and to this solution U, we obtain the equation

$$2\iint_\Delta \left(\frac{\partial U}{\partial x}\right)^2 d\omega + \int_{PQ} U^2\, dx = 0 \quad ,$$

which implies that the second integral is zero, hence $U=0$ on PQ. Q.E.D.

Remark. If we consider a solution $U(x,y)$ defined in a strip $0 \leq y \leq y_0$ and zero at infinity within this strip, then the above argument, in which we take for AC and BD the segments parallel to Oy which are extended indefinitely in both directions, again shows that U is unique. We may now readily assert that the solution (84) tends to zero when x tends to infinity, y constant, when $f(t)$ tends to zero when t increases indefinitely. There then exists a unique solution, given by (84), taking the values $f(z)$ for $y=0$ and zero when x increases indefinitely, y constant $(y \geq 0)$.

PROPERTIES OF THE SOLUTION OF THE EQUATION $E=0$

Let $U(x,y)$ be a solution of the equation (75), $E=0$, which admits continuous first partial derivatives in a domain D.

I. The function $U(x,y)$ admits in D partial derivatives of all orders.

In order to study $U(x,y)$ in a neighborhood of a point $M(X,Y)$ of D, we shall consider a rectangle R, defined by $x_0 < x < x_1$, $y_0 < y < y_1$, contained in D and containing the point M. Let M' be the point with coordinates X, $Y+h$, where h is positive and sufficiently small so as to allow M' to belong to R and let us denote by Δ the part of R corresponding to $y < Y$ (fig. 90). The function of x,y defined by

$$V(x,y,X,Y+h) = \frac{e^{-(x-X)^2/4(Y+h-y)}}{\sqrt{Y+h-y}} \quad ,$$

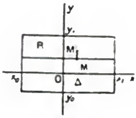

Fig. 90.

for $y < Y+h$, is a solution of the adjoint equation $F(v)=0$ since it is deduced from the solution of $E(u)=0$ envisaged in no. 298, by changing t to X and y to $-y+Y+h$.

It admits continuous derivatives in Δ and on its boundary Γ. When we apply (88) to the rectangle Δ by taking $u=V$ and $v=U$, the double integral is zero and we obtain

$$\int_{x_0}^{x_1} U(x,Y)\,\frac{e^{-(x-X)^2/4h}}{\sqrt{h}}\,dx = \int_{x_0}^{x_1} U(x,y_0)V(x,y_0,X,Y+h)\,dx$$
$$+\int_{y_0}^{Y} [w(x_1,y,X,Y+h) - w(x_0,y,X,Y+h)]\,dy$$

with

$$w(x,y,X,Y) = \left[\frac{1}{2}\,U(x,y)\,\frac{x-X}{Y-y} + \frac{\partial U}{\partial x}(x,y)\right] V(x,y,X,Y) \quad .$$

Following the proposition which was proved in no. 298, the first member of this equality tends to $2\sqrt{\pi}\,U(X,Y)$ when h tends towards zero. In the second member, the first integral has as its limit the expression obtained by setting $h = 0$, which is clear. In the second integral of the second member, we have an apparent discontinuity of the function w, for $y = Y$, when we set $L = 0$, but owing to the presence of the exponential factor of V, there is again uniform convergence towards $w(x,y,X,Y)$. We then arrive at the definitive result

$$2\sqrt{\pi}\,U(X,Y) = \int_{x_0}^{X} U(x,y_0)V(x,y_0,X,Y)\,dx$$
$$+\int_{y_0}^{Y} [w(x_1,y,X,Y) - w(x_0,y,X,Y)]\,dy \quad , \tag{90}$$

analogous to the formula (59) of no. 293.

Following the result of no. 298, the first integral of the second member is an analytic function of X,Y, where the radius of convergence relative to X is infinite. In the second integral there appear terms of the form

$$\frac{T(y)}{(Y-y)^m}\; e^{(x-X)^2/4(Y-y)}\,dy \quad , \qquad x = x_1 \text{ or } x_0 \quad , \tag{91}$$

where m is equal to $1/2$ or $3/2$, and we must integrate up to Y; $T(y)$ is continuous, hence $|T(y)| < M$.

If we replace X by $X+\xi+i\eta$, we obtain a holomorphic function of $\xi+i\eta$ provided that the real part of

$$(x - X + \xi + i\eta)^2$$

is greater than a fixed positive number.

For the convergence of the integral taken from y_0 to $Y-\varepsilon$, an integral which is a holomorphic function of $\xi+i\eta$, will be uniform. It is sufficient to have $2|\xi+i\eta|$ less than the smallest of the two numbers x_1-X and $X-x_0$ in order for this to be the case. It follows that the second integral in (90) defines an analytic function of X, for any Y in R. Moreover, if X is replaced by $X+\xi+i\eta$, we obtain a function of $\xi+i\eta$ which is certainly holomorphic in a circle of radius ρ independent of Y and whose modulus is bounded in this circle by a number K independent of Y. K is a sum of integrals of the form

$$M\int_0^c \frac{e^{-\tau/t}}{t^m}\,dt \quad , \qquad m=\frac{1}{2} \ , \qquad m=\frac{3}{2} \qquad \tau>0 \ .$$

On the other hand, the integral of an expression of the form (91) taken between y_0 and Y, converges uniformly thanks to the exponential function appearing. It follows that the second integral in (90) admits derivatives of all orders with respect to Y and that each of its derivatives is a regular analytic function of X. The function $U(X,Y)$ admits partial derivatives of all orders. Moreover,

II. $U(X,Y)$ *and each of its derivatives with respect to* Y *are regular analytic functions of* X.

THE HOLMGREN CONDITIONS

We have seen that on any point of a segment parallel to the axes of y, $x=X$, $U(X+\xi+i\eta,Y)$ is holomorphic for $|\xi+i\eta|<\rho$ and that its modulus is bounded by a fixed number K. (Equivalently we can say that these functions are analytic at $\xi+i\eta$.) Following the Cauchy inequalities, we will have on this segment

$$\left|\frac{\partial^n U}{\partial X^n}\right| \rho^n \leqslant Kn[\ , \qquad\qquad n=0,1,2,\dots \ . \tag{92}$$

But for U a solution of $E(U)=0$ and having derivatives of all orders, we have

$$\frac{\partial^2 U}{\partial Y^2} = \frac{\partial^2}{\partial X^2}\left(\frac{\partial U}{\partial Y}\right) = \frac{\partial^4 U}{\partial X^4} \quad ,$$

and more generally,

$$\frac{\partial^n U}{\partial Y^n} = \frac{\partial^{2n} U}{\partial X^{2n}} \quad .$$

Now utilizing the inequality (92), we obtain

$$\left|\frac{\partial^n U}{\partial Y^n}\right| < \frac{K(2n)!}{\rho^{2n}} \quad , \qquad n = 0,1,2,\ldots \quad . \tag{93}$$

(More generally, we would obtain bounds for the derivatives taken n-times with respect to X and p times with respect to Y, which are defined in a domain completely within D where the solution is defined. The conditions (93) are due to Holmgren; the derivatives satisfy analogous conditions. *These necessary conditions are also sufficient in order for the Cauchy problem to be solved in a neighborhood of a parallel segment to* Oy *when* u *and* $\partial u/\partial x$ *are given on this segment.*

Effectively, if we consider two infinitely-differentiable functions on a segment $y_0 \leq y \leq y_1$ and whose derivatives satisfy condition (93) with differing values of K and ρ, then by taking the largest of the two numbers K and the smallest of the two numbers ρ, we can assume that these numbers are the same. The series (77) is then convergent for any y on this segment for $|x-x_0| \leq \rho' < \rho$, and defines a solution of equation (75) for these values of x and y.

UNIQUENESS

There exists a single continuous solution along with its first partial derivatives in a rectangle with sides parallel to the axes and taking, along with its partial derivative with respect to x, *the given values on a segment* $x = x_0 = \text{const.}$ *of this rectangle.*

For, if this solution and this derivative are zero on this segment, then we have

$$\frac{\partial^{2n} u}{\partial x^{2n}} = \frac{\partial^n u}{\partial y^n} = 0 \quad , \qquad \frac{\partial^{2n+1} u}{\partial x^{2n+1}} = \frac{\partial^n}{\partial y^n}\left(\frac{\partial u}{\partial x}\right) = 0$$

at every point x_0, y of this segment, and the solution, analytic for $y = \text{const.}$ is zero on this line, since it is zero along with its derivatives at the point x_0.

HOLMGREN FUNCTIONS

An infinitely differentiable function $\phi(y)$ on a segment $\gamma \leq y \leq \delta$ will be called a Holmgren function, or *function* H, if its derivatives satisfy a condition of the form (93). The quasi-analytic classes of Denjoy (I, 80) for which

$$[\phi^{(n)}(y)]^{1/n} < kn(\log n) \ldots (\log_q n) \qquad n = 0,1,2,\ldots$$

are such functions H. But the class of functions H contains functions

which are not quasi-analytic. Such functions H are not defined simply by prescribing their value and the values of their derivatives at a point of the segment where they are defined; *they might even coincide in an interval without being identical.* Let us consider the function

$$U(x,y) = \int_{\gamma}^{y} \theta(Y) \frac{e^{-x^2/4(y-Y)}}{\sqrt{y-Y}} dY \quad ,$$

where $\theta(Y)$ is continuous on (γ,δ) and $\gamma \leqslant y \leqslant \delta$. We can easily verify that $U(x,y)$ which is defined for $\gamma \leqslant y \leqslant \delta$ and $x \geqslant 0$, satisfies equation (75) in the interior of this half-strip. For $x = x_0 > 0$, we obtain a function $\phi(y)$ which is a function H for $\gamma < \gamma \leqslant y \leqslant \delta' < \delta$. If we replace $\theta(Y)$ by $\theta(Y) + \omega(Y)$, $\omega(Y)$ being continuous, zero for $\gamma \leqslant Y \leqslant \eta'$ and positive for $\eta' < Y \leqslant \delta$, where $\gamma' < \eta' < \delta'$, we obtain another Holmgren function for $\gamma' \leqslant y \leqslant \delta'$, which coincides with $\phi(y)$ when y belongs to the segment (γ',η') and is greater than $\phi(y)$ for $\eta' < y \leqslant \delta'$.